Stat 400 Midterm 2 Spring 2012

1.

$X \backslash Y$	0	1	2	
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$5/8$
1	$\frac{1}{4}$	0	$1/8$	$3/8$
	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	

(a)

x	0	1
$P(X=x)$	$\frac{5}{8}$	$\frac{3}{8}$

$E(X) = \frac{3}{8}$ $\quad E(X^2) = \frac{3}{8}$

$V(X) = E(X^2) - E(X)^2$

$= \frac{3}{8} - \frac{9}{64} = \frac{15}{64}$

$\sigma_X = \frac{\sqrt{15}}{8}$

y	0	1	2
$P(Y=y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

$E(Y) = \frac{3}{4}$ $\quad E(Y^2) = \frac{5}{4}$

$V(Y) = E(Y^2) - E(Y)^2$

$= \frac{5}{4} - \frac{9}{16} = \frac{11}{16}$

$\sigma_Y = \frac{\sqrt{11}}{4}$

(b) No $\quad P(X=1, Y=1) \neq P(X=1)\,P(Y=1)$

$P(X=1, Y=1) = 0$ and $P(X=1)\,P(Y=1) = \left(\frac{1}{4}\right)\left(\frac{3}{8}\right) = \frac{3}{32}$

(c) $P(W=0) = P(X+Y=0) = P(X=0, Y=0) = \frac{1}{4}$

$P(W=1) = P(X+Y=1) = P(X=0, Y=1) + P(X=1, Y=0)$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(W=2) = P(X+Y=2) = P(X=0, Y=2) + P(X=1, Y=1)$

$= \frac{1}{8} + 0 = \frac{1}{8}$

$P(W=3) = P(X+Y=3) = P(X=1, Y=2) = \frac{1}{8}$

w	0	1	2	3
$P(W=w)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$

) $E(XY) = 0 + 0 + 0$
$+ 0 + 0 + (1)(2)(\frac{1}{8})$

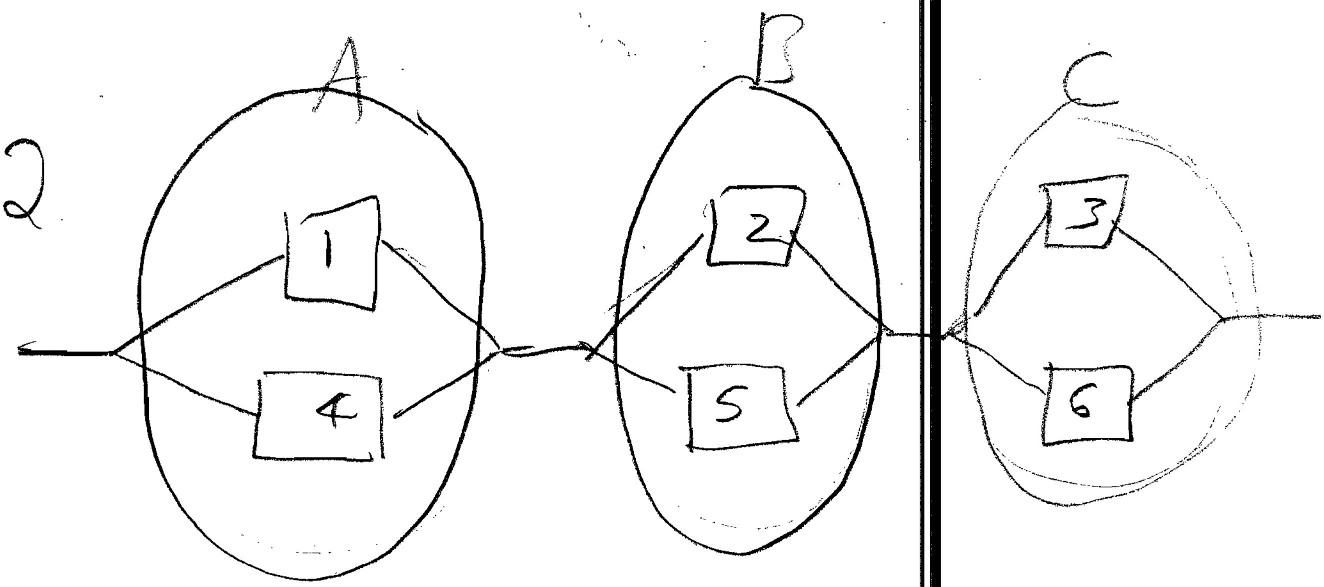
$= \frac{2}{8}$

$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{2}{8} - \left(\frac{3}{8}\right)\left(\frac{3}{4}\right)$

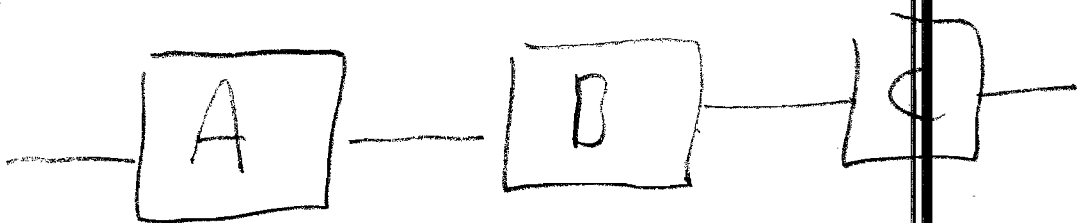
$= \frac{8}{32} - \frac{9}{32} = -\frac{1}{32}$

$= \frac{Cov(X,Y)}{\ldots} = \frac{\left(-\frac{1}{32}\right)}{\sqrt{15}\ \sqrt{11}} = \frac{-1}{\sqrt{(15)(11)}} = \frac{-1}{\sqrt{165}}$

2



Define a new system and new random variables Y_1, Y_2, Y_3, Y by



and

$(Y_1 = t) = (A \text{ fails at time } t)$

$(Y_2 = t) = (B \text{ fails at time } t)$

$(Y_3 = t) = (C \text{ fails at time } t)$

$(Y = t) = \left(\begin{array}{c} \text{the new system} \\ \text{fails at time } t \end{array} \right).$

Hence

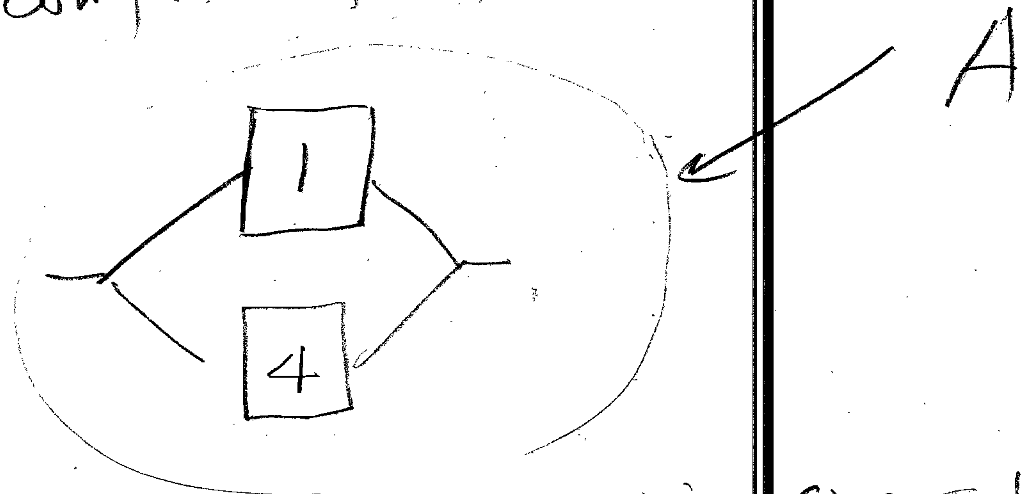
$$(Y>t) = (Y_1>t) \cap (Y_2>t) \cap (Y_3>t)$$

$$P(Y>t) = P(Y_1>t) \cdot P(Y_2>t)\, P(Y_3>t)$$

$$= P(Y_1>t)^3$$

(since A, B and C are identical)

We compute $P(Y_1>t)$



$$(Y_1 \le t) = (X_1 \le t) \cap (X_4 \le t)$$

$$P(Y_1 \le t) = P(X_1 \le t)\, P(X_2 \le t)$$

$$= \left(1-e^{-\lambda t}\right)^2$$

$$= 1-2e^{-\lambda t}+e^{-2\lambda t}$$

$$P(Y_1>t) = 1-(1-2e^{-\lambda t}+e^{-2\lambda t})$$

So

$$P(Y_1 > t) = 2e^{-\lambda t} - e^{-2\lambda t}$$

$$P(Y > t) = \left(2e^{-\lambda t} - e^{-2\lambda t}\right)^3$$

Hence since $X = Y$ by construction

$$P(X > t) = P(Y > t)$$

$$P(X > t) = \left(2e^{-\lambda t} - e^{-2\lambda t}\right)^3$$

$$P(X \le t) = 1 - P(X > t)$$

$$= 1 - \left(2e^{-\lambda t} - e^{-2\lambda t}\right)^3$$

3 (a)

We have

X \ Y	0	1	
0			q
1			p
	q	p	

$(q = 1-p)$

Since X and Y are independent will fill in the matrix by taking products

X \ Y	0	1	
0	q^2	pq	q
1	pq	p^2	p
	q	p	

This solves (a)

(b) W takes values

$$P(W=0) = P(X+Y=0) = P(X=0, Y=0) = q^2$$

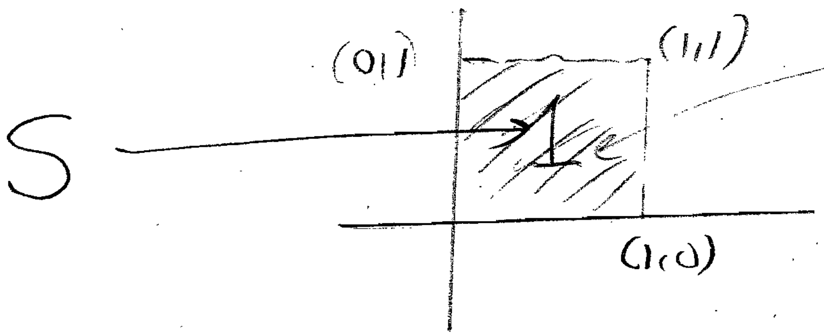
$$P(W=1) = P(X+Y=1) = P(X=0, Y=1) + P(X=1, Y=0) = 2pq$$

$$P(W=2) = P(X+Y=1) = P(X=1, Y=1) = p^2$$

(c) $W \sim \text{Bin}(2, p)$ (2 flips of a fair coin with head probability p)

4. (a) $f_{X,Y}(x,y) = \begin{cases} 1, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

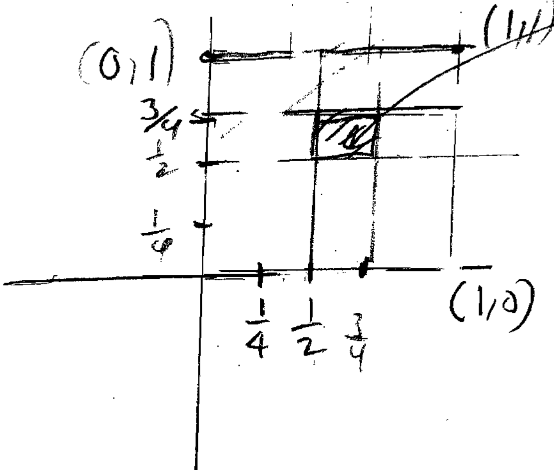
(So (X,Y) has uniform distribution on the unit square S)



$f_{X,Y}(x,y)$ is 1 here and zero outside S.

Hence if R is a region in the plane
$P((X,Y) \in R) = \text{Area}(R \cap S)$

(b)



$R = \frac{1}{4}$ by $\frac{1}{4}$ rectangle contained in the unit square

$R \subset S$ so $R \cap S = R$

$\text{Area}(R) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}$

$P\left(\frac{1}{2} \le X \le \frac{3}{4}, \frac{1}{2} \le Y \le \frac{3}{4}\right) = \frac{1}{16}$

5. Let $Y = X^2$ so $h(x) = x^2$

The support of $f_X(x)$ is $[0,1]$.

The support of $f_Y(y)$ is $[h(0), h(1)] = [0,1]$

so $f_Y(y) = 0$ unless $0 \le y \le 1$.

We compute the inverse of $y = h(x) = x^2$

$y = x^2$, $x = \sqrt{y}$ so $g(y) = \sqrt{y}$

Finally we apply the Engineer's Way to compute $f_Y(y)$: We assume $0 \le y \le 1$

$$f_Y(y)\,dy = f_X(x)\Big|_{x=\sqrt{y}} \; dx\Big|_{x=\sqrt{y}}$$

$$= 2x\Big|_{x=\sqrt{y}} \, d\sqrt{y} \qquad \left(d(\sqrt{y}) = \frac{1}{2\sqrt{y}}\,dy \right.$$

$$= 2\sqrt{y}\,\frac{1}{2\sqrt{y}}\,dy$$

$$= 1\,dy$$

$$f_Y(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

so $Y \sim U(0,1)$