Solutions:

#1.

(a)

X \ Y	0	1	2	$p_X(x)$
0	1/16	1/16	1/8	$\frac{2}{8}$
1	1/8	1/4	1/8	$\frac{1}{2}$
2	1/16	1/16	1/8	$\frac{2}{8}$
$p_Y(y)$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	

(b) Not independent.

$$P(X=1, Y=1) = \frac{1}{4}$$

$$\neq P(X=1)\cdot P(Y=1) = \frac{1}{2}\cdot\frac{3}{8}$$

(c)

$W=0 \sim X=0, Y=0$ $\quad \frac{1}{16}$ $\quad \frac{1}{16}$

$W=1 \sim X=0, Y=1$ $\quad \frac{1}{16}$; $X=1, Y=0$ $\quad \frac{1}{8}$ $\quad \rightarrow \frac{3}{16}$

$W=2$: $X=0, Y=2$ $\quad \frac{1}{8}$; $X=2, Y=0$ $\quad \frac{1}{16}$; $X=1, Y=1$ $\quad \frac{1}{4}$ $\quad \rightarrow \frac{7}{16}$

$W=3$: $X=1, Y=2$ $\quad \frac{1}{8}$; $X=2, Y=1$ $\quad \frac{1}{16}$ $\quad \rightarrow \frac{3}{16}$

$W=4$: $X=2, Y=2$ $\quad \frac{1}{8}$ $\quad \frac{2}{16}$

W	0	1	2	3	4
p(w)	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{7}{16}$	$\frac{3}{16}$	$\frac{2}{16}$

(d) $E[X] = 0 \cdot \frac{2}{8} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{2}{8} = 1$

$E[Y] = 0 \cdot \frac{2}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} = \frac{9}{8}$

$$E[XY] = 0 \cdot \left[\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16}\right] + 1 \cdot 1 \cdot \frac{1}{4} + 1 \cdot 2 \cdot \frac{1}{8}$$

$$+ 2 \cdot 1 \cdot \frac{1}{16} + 2 \cdot 2 \cdot \frac{1}{8}$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{2} = \frac{9}{8}$$

$Cov[X, Y] = E[XY] - E[X] \cdot E[Y] = 0$

(e) $\rho_{XY} = \frac{Cov[X, Y]}{\sqrt{Var[X] \cdot Var[Y]}} = 0$

#2 $L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta^2} \cdot x_i e^{-\frac{x_i}{\theta}} = \frac{1}{\theta^{2n}} \prod_{i=1}^{n} x_i \, e^{-\frac{\sum_i x_i}{\theta}}$

$$\ell(\theta) = \log L(\theta) = -2n \log \theta + \sum_{i=1}^{n} \log x_i - \frac{\sum_i x_i}{\theta}$$

$$\frac{\partial}{\partial \theta} \ell(\theta) = -\frac{2n}{\theta} + \frac{\sum_i x_i}{\theta^2} = 0 \quad \Rightarrow \quad \hat{\theta} = \frac{\sum_i x_i}{2n}$$

$$\frac{d^2}{d\theta^2} \ell(\theta) = \frac{2n}{\theta^2} - \frac{2 \sum_i x_i}{\theta^3}$$ at $\hat{\theta}$ we have

$$\frac{d^2}{d\theta^2} \ell(\theta) \Big|_{\theta = \hat{\theta}} = \frac{2n}{(\sum_i x_i)^2} - \frac{16n^3}{(\sum_i x_i)^2} < 0$$

$\Rightarrow \hat{\theta}$ is the maximum of $\ell(\theta)$.

$$E[\hat{\theta}] = \frac{E[\sum_i x_i]}{2n} = \frac{n \cdot 2\theta}{2n} = \theta$$ unbiased!

#3 $E[\bar{X}] = E[\frac{1}{n}\sum_{i=1}^{n} x_i]$

$= \frac{1}{n} \cdot n \cdot E[X]$

$= \mu$

So, it's unbiased.

#4 $E[X] = 50$ $\quad Var[X] = 100 \Rightarrow \sigma_X = 10$

$P(X \geq 60) = P\left(\frac{X-50}{10} \geq \frac{60-50}{10}\right)$

$= P\left(\frac{X-50}{10} \geq 1\right)$

$\approx P(Z \geq 1) \qquad Z \sim N(0,1)$

$= 1 - \Phi(1)$

check the table and find this value.