Stat 400

Practice Final Solutions

1.

x	1	2	3
$P(X=x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(a) $E(X) = (1)(\frac{1}{3}) + (2)(\frac{1}{3}) + (3)(\frac{1}{3}) = \frac{1+2+3}{3} = 6/3 = 2$

(b) $E(X^2) = (1)^2(\frac{1}{3}) + (2)^2(\frac{1}{3}) + (3)^2(\frac{1}{3}) = \frac{1+4+9}{3} = 14/3$

(c) $V(X) = E(X^2) - E(X)^2 = \frac{14}{3} - \left(\frac{6}{3}\right)^2$

$= \frac{14}{3} - \frac{36}{9} = \frac{42}{9} - \frac{36}{9} = \frac{6}{9} = \frac{2}{3}$

(d)

$$F(x) = \begin{cases} 0, & x<1 \\ \frac{1}{3}, & 1 \le x < 2 \\ \frac{2}{3}, & 2 \le x < 3 \\ 1, & 3 \le x \end{cases}$$

old values

(e) Y takes values $1-1, 2-1, 3-1 = 0, 1, 2$ each with probability $\frac{1}{3}$

2. To save effort, I am going to use the formula I told you about (a, b positive integers)

$$(*) \quad \int_0^1 x^a (1-x)^b \, dx = \frac{a!\, b!}{(a+b+1)!}$$

I expected you to do the integral directly.

(a) $$E(X) = \int_0^1 x\, 2(1-x)\, dx = 2\int_0^1 x^1 (1-x)^1 \, dx$$

$$\overset{\text{by } (*)}{=} 2\,\frac{(1!)(1!)}{3!} = \frac{2}{6} = \frac{1}{3}$$

(b) $$E(X^2) = \int_0^1 x^2\, 2(1-x)\, dx = 2\int_0^1 x^2 (1-x)^1 \, dx$$

$$\overset{\text{by } (*)}{=} 2\,\frac{(2!)(1!)}{4!} = \frac{(2)(2)(1)}{24} = \frac{1}{6}$$

$$V(X) = E(X^2) - E(X)^2 = \frac{1}{6} - \frac{1}{9}$$

$$= \frac{3}{18} - \frac{2}{18} = \frac{1}{18}$$

(c) For (c) we need (for $0 \le \cancel{x} \le 1$)

$$F(x) = \int_0^x 2(1-t)\, dt = 2\int_0^x dt - 2\int_0^x t\, dt$$

So $F(x) = 2x - 2\left(\frac{x^2}{2}\right) = 2x - x^2$

so

$$F(x) = \begin{cases} 0, & x \le 0 \\ 2x - x^2, & 0 \le x \le 1 \\ 1, & 1 \le x \end{cases}$$

(d) To find the median $\tilde{\mu}$ we have to solve the equation $F(\tilde{\mu}) = \frac{1}{2}$ for $\tilde{\mu}$

So we have to solve

$$2\tilde{\mu} - \tilde{\mu}^2 = \frac{1}{2}$$

$$\Rightarrow \tilde{\mu}^2 - 2\tilde{\mu} + \frac{1}{2} = 0 \Rightarrow \tilde{\mu} = \frac{2 \pm \sqrt{4-2}}{2} = 1 \pm \frac{1}{\sqrt{2}}$$

But clearly $0 < \tilde{\mu} < 1$ so we have to choose the (smaller) root $\tilde{\mu} = 1 - \frac{1}{\sqrt{2}}$

(because $1 + \frac{1}{\sqrt{2}} > 1$, $F(1 + \frac{1}{\sqrt{2}}) = 1$ so not $\frac{1}{2}$)

(e) Let $\eta(.75)$ denote the 75th percentile. I will abbreviate this to η. We have to solve

$$2\eta - \eta^2 = \frac{3}{4} \Rightarrow \eta^2 - 2\eta + \frac{3}{4} = 0$$

$$\Rightarrow \eta = \frac{2 \pm \sqrt{4-3}}{2} = \frac{2 \pm 1}{2} = 3/2 \text{ or } \frac{1}{2}$$

Once again it is clear we have to choose $\eta = \frac{1}{2}$.

because $F(3/2) = 1$ so $3/2$ is <u>NOT</u> a solution of the equation

$$F(\eta) = \frac{3}{4}$$

3. Let X be the event that at least one couple sits together

Let A = Jack sits beside Jill

B = Dick sits beside Jane

so

$$X = A \cup B$$

The total number of ways the 4 people can be arranged is $4! = 24$

so

$$P(X) = \frac{\#(X)}{24} \qquad (*)$$

We have

$$\#(X) = \#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$$

Clearly $\#(A) = \#(B)$ so

$$\underline{\#(X) = 2\#(A) - \#(A \cap B)} \qquad (**)$$

So we have to compute

(1) $\#(A)$

(2) $\#(A \cap B)$

1. Computation of $\#(A)$

The trick is to "glue Jack and Jill together" to form Jack Jill

So
$$\#(A) = \underset{\uparrow \text{ exchange Jack and Jill}}{2}\left(\begin{array}{l}\text{\# of arrangements of } \underline{3} \text{ "people"} \\ \text{Jack Jill, Dick, Jane}\end{array}\right)$$

$$= 2(3!) = 12$$

2. Computation of $\#(A \cap B)$

Now we glue Jack and Jill together (Jack Jill) and Dick and Jane together (Dick Jane)

So
$$\#(A \cap B) = \underset{\text{exchange } J \& J}{(2)}\ \underset{\text{exchange } D \& J}{(2)}\left(\begin{array}{l}\text{\# of arrangements of 2} \\ \text{"people" } J \& J \quad D \& J\end{array}\right)$$

$$= (2)(2)(2) = 8$$

Plugging into $(**)$ we get
$$\#(X) = 2(12) - 8 = 16$$

and into $(*)$ we get $P(X) = \frac{16}{24} = \frac{2}{3}$; (the same as for 3 couples!?)

4.

$x \backslash y$	0	1	
0	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
	$\frac{1}{4}$	$\frac{3}{4}$	

(a)

x	0	1
$P(X=x)$	$\frac{1}{4}$	$\frac{3}{4}$

so $E(X) = \frac{3}{4}$, $E(X^2) = \frac{3}{4}$

$\Rightarrow V(X) = \frac{3}{4} - \frac{9}{16} = \frac{3}{16}$

$\Rightarrow \sigma_X = \frac{\sqrt{3}}{4}$

y	0	1
$P(Y=y)$	$\frac{1}{4}$	$\frac{3}{4}$

$\xRightarrow{\text{same as for } X}$ $E(Y) = \frac{3}{4}$, $\sigma_Y = \frac{\sqrt{3}}{4}$

(b) No

(c)

z	0	1	2
$P(Z=z)$	0	$\frac{1}{2}$	$\frac{1}{2}$

(d) $Cov(X,Y) = E(XY) - E(X)E(Y)$

$= (1)(1)(\frac{1}{2}) - (\frac{3}{4})(\frac{3}{4})$

$= \frac{1}{2} - \frac{9}{16} = -\frac{1}{16}$

(e) $\rho_{XY} = \dfrac{(-\frac{1}{16})}{\frac{\sqrt{3}}{4}\,\frac{\sqrt{3}}{4}} = \dfrac{(-\frac{1}{16})}{(\frac{3}{16})} = \dfrac{-1}{3}$

5(1) We will cover this material on Tuesday

The formula (*) for the "lower-tailed" confidence interval for μ with σ^2 unknown is on page 290 of the text ("confidence upper bound) on Lecture 29, Theorem 3, of my Handwritten Notes. Here we have

$\alpha = .05$ (since $1-\alpha = .95$)

$n = 26$ so $n-1 = 25$

$t_{\alpha, n-1} = t_{.05, 25} = 1.708$ (from text Tab. A5, page A-9)

$\bar{x} = 24.36$, $s = 370.69$

So (formula (*))

$$\left(-\infty, \bar{x} + t_{\alpha, n-1} \frac{s}{\sqrt{n}}\right)$$

$$= \underbrace{\left(-\infty, (24.36) + (1.708)\left(\frac{370.69}{\sqrt{26}}\right)\right)}_{\text{leave it like this since you won't have a calculator}}$$

5 (ii) The formula for the two-sided prediction interval is in the text, pg. 292, however the text does not give the formula you need here. The upper-tailed prediction interval is Theorem 3 of Lecture 32 of my notes. The lower-tailed prediction interval is

$$\left(-\infty,\ \bar{x} + t_{\alpha, n-1}\sqrt{\frac{n+1}{n}}\, s\right)$$

.05   25

$$= \underbrace{\left(-\infty,\ 24.36 + (1.708)\sqrt{\frac{27}{26}}\,(370.69)\right)}_{\text{leave it like this}}$$

We may not cover this material this year, I don't know yet.

6. Again we will cover this on Tuesday. I should have given

<u>Theorem A</u> The random variable

$$T = \frac{\overline{X} - \mu}{\left(S/\sqrt{n}\right)}$$ has t-distribution (*)

with $n-1$ degrees of freedom.

Now we can prove the formula for the upper-tailed t-confidence interval - see Section 4 of Lecture 29 of my Handwritten Notes. This was a HW problem to get ready for the final

<u>Theorem B</u> $\left(\overline{X} - t_{\alpha, n-1}\frac{S}{\sqrt{n}}, \infty\right)$

is a $100(1-\alpha)\%$ confidence interval for μ (from a normal population)

Proof We are required to prove

$$P\left(\mu \in \left(\overline{X} - t_{\alpha, n-1}\frac{S}{\sqrt{n}}, \infty\right)\right) = 1-\alpha$$

$$\text{LHS} = P\left(\overline{X} - t_{\alpha, n-1}\frac{S}{\sqrt{n}} < \mu\right)$$

(we have to get T_{n-1} from (※) on pg 9)

swap

$$= P\left(\overline{X} - \mu < t_{\alpha, n-1}\frac{S}{\sqrt{n}}\right)$$

divide

from Theorem A

$$= P\Bigg(\underbrace{\frac{\overline{X} - \mu}{(S/\sqrt{n})}}_{T_{n-1}} < t_{\alpha, n-1}\Bigg) = \text{this area}$$

$= 1-\alpha$ from the picture