Lecture 14 : The Gamma Distribution and its Relatives

The gamma distribution is a continuous distribution depending on two parameters, α and β. It gives rise to three special cases

1. The exponential distribution ($\alpha = 1, \beta = \dfrac{1}{\lambda}$)
2. The r-Erlang distribution ($\alpha = r, \beta = \dfrac{1}{\lambda}$)
3. The chi-squared distribution ($\alpha = \dfrac{\nu}{2}, \beta = 2$)

The Gamma Distribution

Definition

A continuous random variable X is said to have gamma distribution with parameters α and β, both positive, if

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \textit{otherwise} \end{cases}$$

What is $\Gamma(\alpha)$?
$\Gamma(\alpha)$ is the gamma function, one of the most important and common functions in advanced mathematics. If α is a positive integer n then

$$\Gamma(n) = (n-1)!$$

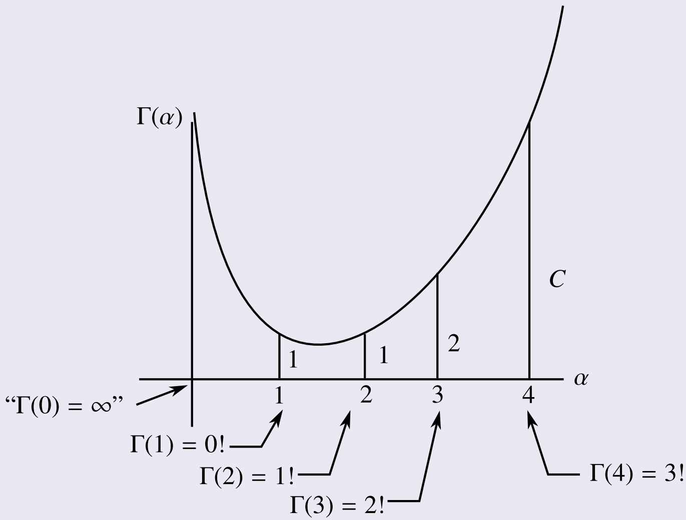
(see page 17)

Definition (Cont.)

So $\Gamma(\alpha)$ is an interpolation of the factorial function to all real numbers.

Z $\lim_{\alpha \to 0} \Gamma(\alpha) = \infty$

Graph of $\Gamma(\alpha)$



I will say more about the gamma function later. It isn't that important for Stat 400, here it is just a constant chosen so that

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

The key point of the gamma distribution is that it is of the form

$$\text{(constant) (power of } x\text{) } e^{-cx}, c > 0.$$

The r-Erlang distribution from Lecture 13 is almost the most general gamma distribution.

The only special feature here is *that* α *is a whole number* r*.*
Also $\beta = \dfrac{1}{\lambda}$ where λ is the Poisson constant.

Comparison Gamma distribution

$$\left(\frac{1}{\beta}\right)^{\alpha} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

r-Erlang distribution $\alpha = r, \beta = \dfrac{1}{\lambda}$

$$\lambda^{r} \frac{1}{(r-1)!} x^{r-1} e^{-\lambda x}$$

Proposition

Suppose X has gamma distribution with parameters α and β then

(i) $E(X) = \alpha\beta$

(ii) $V(X) = \alpha\beta^2$

so for the r-Erlang distribution

(i) $E(X) = \dfrac{r}{\lambda}$

(ii) $V(X) = \dfrac{r}{\lambda^2}$

Proposition (Cont.)

As in the case of the normal distribution we can compute general gamma probabilities by standardizing.

Definition

A gamma distribution is said to be standard if $\beta = 1$. Hence the pdf of the standard gamma distribution is

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The cdf of the standard

Definition (Cont.)

gamma function is called the incomplete gamma function (divided by $\Gamma(\alpha)$)

$$F(x) = \frac{1}{\Gamma(\alpha)} \int_0^x x^{\alpha-1} e^{-x} dx$$

(see page 13 for the actual gamma function)
It is tabulated in the text Table A.4 for some (integral values of α)

Proposition

Suppose X has gamma distribution with parameters α and β. Then $Y = \dfrac{X}{\beta}$ has standard gamma distribution.

Proof.

We can prove this, $Y = \dfrac{X}{\beta}$ so $X = \beta y$.

Now $f_X(x)dx = \dfrac{1}{\beta^\alpha}\dfrac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}dx$.

Now substitute $x = \beta y$ to get

$$
\begin{aligned}
f_Y(y)dy &= \frac{1}{\beta^\alpha}\frac{1}{\Gamma(\alpha)}(\beta y)^{\alpha-1}e^{-\frac{\beta y}{\beta}}d(\beta y) \\
&= \frac{1}{\cancel{\beta^\alpha}}\frac{1}{\Gamma(\alpha)}\cancel{\beta^{\alpha-1}}y^{\alpha-1}e^{-y}\cancel{\beta}dy \\
&= \underbrace{\frac{1}{\Gamma(\alpha)}y^{\alpha-1}e^{-t}}_{\text{standard gamma}}dy
\end{aligned}
$$

□

Example 4.24 (cut down)

Suppose X has gamma distribution with parameters $\alpha = 8$ and $\beta = 15$. Compute

$$P(60 \leq X \leq 120)$$

Solution

Standardize, divide EVERYTHING by $\beta = 15$.

$$\begin{aligned} P(60 \leq X \leq 120) &= P\left(\frac{60}{15} \leq \frac{X}{15} \leq \frac{120}{15}\right) \\ &= P(4 \leq Y \leq 8) = F(8) - F(4) \\ \textit{from table A.4} & \\ &= .547 - .051 = .496 \end{aligned}$$

The Chi-Squared Distribution

Definition

Let ν (Greek letter nu) be a positive real number. A continuous random variable X is said to have chi-squared distribution with ν degrees of freedom if X has gamma distribution with $\alpha = \nu/2$ and $\beta = 2$. Hence

$$f(x) = \begin{cases} \dfrac{1}{2^{\nu/2}} \Gamma(\nu/2)^{x^{\nu/2-1}} e^{-x/2}, x > 0 \\ 0, \quad \textit{otherwise.} \end{cases}$$

We will write $X \sim \chi^2(\nu)$.

capital chi

The reason the chi-squared distribution is that if

$$Z \sim N(0,1) \quad \text{then} \quad X = Z^2 \sim \chi^2(1)$$

and if $Z_1, Z_2, \ldots, Z_m$ are independent random variables the

$$Z_1^2 + Z_2^2 + \cdots Z_m^2 \sim \chi^2(m)$$

(later).

Proposition (Special case of pg. 6)

If $X \sim \chi^2(\nu)$ then

(i) $E(X) = \nu$

(ii) $V(X) = 2\nu$

Appendix : The Gamma Function

Definition

For $\alpha > 0$, the gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

Remark 1

It is more natural to write

this is the variable

$$\Gamma(\alpha) = \int_0^\infty x^{@} e^{-x} \frac{dx}{x}$$

but I won't explain why unless you ask.

Remark 2

In the complete gamma function we integrate from 0 *to infinity whereas for the incomplete gamma function we integrate from* 0 *to* $\underline{x}$*.*

$$F(x;\alpha)\int_0^x y^{\alpha-1}e^{-y}dx.$$

Thus

$$\lim_{x\to\infty} F(x;\alpha) = \Gamma(\alpha).$$

Remark 3

Many of the "special functions" of advanced mathematics and physics e.g. Bessel functions, hypergeometric functions... arise by taking an elementary function of x depending on a parameter (or parameters) and integrating with respect to x leaving a function of the parameter. Here the elementary function is $x^{\alpha-1}e^{-x}$. We "integrate out the x" leaving a function of α.

Lemma

$\Gamma(1) = 1$

Proof.

$$\Gamma(1) = \int_0^\infty e^{-x} dx = (-e^{-x})\Big|_0^\infty = 1 \qquad \square$$

The Functional Equation for the Gamma Function

Theorem

$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \quad \alpha > 0$

Proof.

Integrate by parts

$$\Gamma(\alpha + 1) = \int_0^\infty \overset{u}{x^{-\alpha}} \overset{dv}{e^{-x} dx}$$

$$= (\underset{u}{-x^\alpha} \underset{v}{e^{-x}})\Big|_0^\infty - \int_0^\infty \overset{du}{\alpha x^{\alpha-1}} (\overset{v}{-e^{-x}}) dx$$

$$= \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx$$

$\square$

Corollary

If n is a whole number

$$\Gamma(n) = (n-1)!$$

Proof.

I will show you $\Gamma(4) = 3Q$

$$\begin{aligned}\Gamma(4) &= \Gamma(3+1) = 3\Gamma(3)\\ &= 3\Gamma(2+1) = (3)(2)\Gamma(2)\\ &= (3)(2)\Gamma(1+1) = (3)(2)(1)F(1)\\ &= (3)(2)(1)\end{aligned}$$

In general you use induction. □

We will need Γ(half integers) e.g. $\Gamma\left(\frac{5}{2}\right)$.

Theorem

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

I won't prove this. Try it.

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

In general

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(1)(3)(5)\dots(2n-1)}{2^n}\sqrt{\pi}$$

For statistics we will need only Γ (integer) = (integer-1)!

and $\Gamma\left(\dfrac{\text{add integer}}{2}\right) = \text{above}$