Lecture 17 : Double Integrals

Some of you have not learned how to do double integrals. In this course you will need to do double integrals over rectangles and I will now explain how to do such calculations.

Partial (Indefinite) Integration

In one variable calculus you learned about the indefinite integral $\int f(x)dx$. The point of the indefinite integral was that it was an inverse of the derivative

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$$

(In fact this is the definition of the) indefinite integral.

So

$$\int x^3 dx = \frac{x^4}{4}$$

and

$$\frac{d}{dx}\frac{x^4}{4} = x^3$$

The indefinite integral is defined only up to an arbitrary constant, “the constant of integration”.
The fundamental theorem of calculus then says that to evaluate the *definite* integral $\int_0^b f(x)dx$ you take *any indefinite* integral, evaluate it at the upper limit b and at the lower limit a and subtract the latter from the former.

Now in two variable calculus you have *two* “partial” derivatives $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ so you have two partial indefinite integrals $\int f(x,y)dx$ and $\int f(x,y)dy$.
So (by definition)

$$\frac{\partial}{\partial x}\left(\int f(x,y)dx\right) = f(x,y)$$
$$\frac{\partial}{\partial y}\left(\int f(x,y)dy\right) = f(x,y).$$

Let’s compute

$$\int (x^2+y^2)dx.$$

The idea is to *tract y as a constant.*

$$\int (x^2+y^2)dx = \int x^2 dx + \int y^2 dx$$
$$= \frac{x^3}{3} + y^2 \int dx$$
$$= \frac{x^3}{3} + y^2 x$$

The partial indefinite integral in x is *defined only up to a function of y.* (because $\frac{\partial}{\partial x} g(y) = 0$).

Let's do a harder one

$$\int \sin xy \; dy = -\frac{1}{x}\cos xy.$$

(check

chain rule

$$\frac{\partial}{\partial y}\left(-\frac{1}{x}\cos xy\right) = -\frac{1}{x}\frac{\partial}{\partial xy}(\cos xy)$$
$$= -\frac{1}{x}(-\sin xy)\frac{\partial}{\partial y}(xy)$$
$$= \sin xy$$

Partial (Definite) Integrals

Once you have the partial indefinite integral you have the partial definite integral

$$\int_1^2 (x^2 + y^2)dx = \left(\frac{x^3}{3} + y^3 x\right)\Big|_{x=1}^{x=2}$$

$$= \left(\frac{8}{3} + 2y^2\right) - \left(\frac{1}{3} + y^2\right)$$

$$= y^2 + \frac{7}{3}$$

The Golden Rule

Treat y as a constant throughout and do the one variable integral with respect to x.

Note *the output is a function of* y.

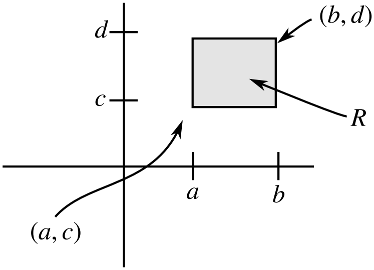
Using Partial Integration to Evaluate Double Integrals over Rectangles

Just as we use the indefinite integral in one variable to evaluate definite integrals in one variable we will use the partial integrals to evaluate (definite) double integrals

$$\int_a^b \int_c^d f(x,y)dx\, dy. \qquad (*)$$

The notation is very confusing because of tradition. Here the x-limits are a and b and the y-limits are c and d *so a and b go with dx and c and d go with dy. Watch out for this later.*

So we have a rectangle R $[a, b] \times [c, d]$ in the xy-plane



When you learn integration theory correctly you will write this integral as

$$\iint\limits_R f(x, y) dx\ dy \qquad (**)$$

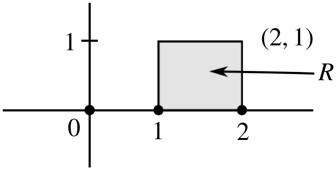
However putting in the limits a, b and c, d is helpful for computations. Think of rewriting (**) as

$$\int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y) dx\ dy.$$

So how do you compute

$$\int_1^2 \int_0^1 (x^2 + y^2)dx\, dy \qquad (**)$$

So the rectangle R is a square



Here is how you compute (**).

There are *two* ways. It is a famous theorem of Fubini that they lead to the same result. We will see this in our example. *Pick the way that leads to the easiest computations.*

First way (do the x-integration first)

1. Write (**) as

$$\int_0^1 \left(\underbrace{\int_1^2 (x^2 + y^2)\,dx}_{\text{do this}} \right) dy$$

$y^2 + {}^7\!/_3$

Z Remember 1 and 2 go with dx and 0 and 1 go with dy (see page 6)

2. Do the inside partial definite integral. The output will be the function of y
$g(y) = y^2 + \dfrac{7}{3}$ (from page 5).

3. Do a one variable definite integral of $g(y)$ with respect to y from 0 to 1.

$$\int_0^1 \left(y^2 + \frac{7}{3}\right) dy = \left(\frac{y^3}{3} + \frac{7}{3}y\right)\Bigg|_{y=0}^{y=0}$$

$$= \frac{1}{3} + \frac{7}{3} = \frac{8}{3}$$

That's it.

The above method is said to evaluate the double integral by iterated one variable integrals.
However the first step is new it is a partial integral with respect to x.

Second way (do the y-integration first)

1. This time we write (**) as

$$\int_1^2 \left(\underbrace{\int_0^1 (x^2 + y^2)\,dy}_{\text{do this first}} \right) dx \qquad x^2 + \frac{1}{3}$$

2. Now we do the partial integration with respect to y so x *is a constant.*

$$\int_0^1 (x^2 + y^2)\,dy = \left(x^2 y + \frac{y^3}{3} \right)\Bigg|_{y=0}^{y=1}$$

$$= x^2 + \frac{1}{3}$$

The output is a function of x.

3. Perform the one variable integration of the output function $h(x) = x^2 + \frac{1}{3}$ with respect to x.

$$\int_1^2 \left(x^2 + \frac{1}{3}\right) dx = \left(\frac{x^3}{3} + \frac{x}{3}\right)\Bigg|_{x=1}^{x=2}$$

$$= \left(\frac{8}{3} + \frac{2}{3}\right) - \left(\frac{1}{3} + \frac{1}{3}\right)$$

$$= \frac{8}{3}$$

So as predicted we got the same answer no matter which order we chose to perform the iterated integrals.

Double Integrals of Product Functions over Rectangles

There is one case in which double integrals one particularly easy to compute.

Definition

Let $f(x,y)$ be a function of two variables x and y. The $f(x,y)$ is a product function if there exist $g(x)$ and $h(g)$ such that

$$f(x,y) = g(x)h(y)$$

Examples

$f(x,y) = e^x \sin y$	*YES (it is a product)*
$f(x,y) = e^x + \sin y$	*NO (it is not a product)*
$f(x,y) = xy$	*YES*
$f(x,y) = x + y$	*NO*

Theorem

$$\int_a^b \int_c^d (g(x)h(y))dx\, dy$$

$$\left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

Z Most functions of x and y are NOT product functions.

Theorem (Cont.)

So

$$\int_0^1 \int_0^1 (xy^2) dx\, dy = \left(\int_0^1 x\, dx\right)\left(\int_0^1 g^2 dy\right)$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{6}$$

So a double integral of a product function over a rectangle is the product of two one variable integrals (one in x, the other in y).