Lecture 29: The confidence interval formulas for the mean in an normal distribution when σ is unknown

1. Introduction

In this lecture we will derive the formulas for the symmetric two-sided confidence interval and the lower-tailed confidence intervals for the mean in a normal distribution *when the variance* σ^2 *is unknown*. At the end of the lecture I assign the problem of proving the formula for the upper-tailed confidence interval. We will need the following theorem from probability theory. Recall that $\overline{X}$ is the sample mean (the point estimator for the populations mean μ) and S^2 is the sample variance, the point estimator for the unknown population variance σ^2. We will need the following theorem from Probability Theory.

Theorem 1

$(\overline{X}\mu)/\dfrac{S}{\sqrt{n}}$ *has t-distribution with* $n-1$ *degrees of freedom.*

2. The two-sided confidence interval formula

Now we can prove the theorem from statistics giving the required confidence interval for μ. Note that it is symmetric around $\overline{X}$. There are also asymmetric two-sided confidence intervals. We will discuss them later. This is one of the basic theorems that you have to learn how to prove.

Theorem 2

The random interval $T = \left(\overline{X} - t_{\alpha/2,n-1}\dfrac{S}{\sqrt{n}}, \overline{X} + t_{\alpha/2,n-1}\dfrac{S}{\sqrt{n}}\right)$ *is a* $100(1-\alpha)\%$*-confidence interval for* μ.

Proof

We are required to prove

$$P\left(\mu \in \left(\overline{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}, \overline{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\right)\right) = 1 - \alpha.$$

We have

Proof (Cont.)

$$\begin{aligned}
\text{LHS} &= P\left(\overline{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} < \mu, \mu < \overline{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\right) \\
&= P\left(\overline{X} - \mu < t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}, -t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} < \overline{X} - \mu\right) \\
&= P\left(\overline{X} - \mu < t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}, \overline{X} - \mu > -t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\right) \\
&= P\left((\overline{X} - \mu)/\frac{S}{\sqrt{n}} < t_{\alpha/2,n-1}, (\overline{X} - \mu)/\frac{S}{\sqrt{n}} > -t_{\alpha/2,n-1}\right) \\
&= P\left(T < t_{\alpha/2,n-1}, T > -t_{\alpha/2,n-1}\right) = P\left(-t_{\alpha/2,n-1} < T < t_{\alpha/2,n-1}\right) = 1 - \alpha
\end{aligned}$$

To prove the last equality draw a picture. □

Once we have an actual sample $x_1, x_2, \ldots, x_n$ we obtain the observed value $\overline{x}$ for the random variable $\overline{X}$ and the observed value s for the random variable S. We obtain the observed value (an ordinary interval) $\left(\overline{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}, \overline{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}\right)$

for the confidence (random) interval $\left(\overline{X} - t_{\alpha/2,n-1}\dfrac{S}{\sqrt{n}}, \overline{X} + t_{\alpha/2,n-1}\dfrac{S}{\sqrt{n}}\right)$ The observed value of the confidence (random) interval is also called the two-sided $100(1-\alpha)\%$ confidence interval for μ.

3. The lower-tailed confidence interval

In this section we will give the formula for the lower-tailed confidence interval for μ.

Theorem 3

The random interval $\left(-\infty, \overline{X} + t_{\alpha,n-1}\dfrac{S}{\sqrt{n}}\right)$ *is a* $100(1-\alpha)\%$*-confidence interval for* μ.

Proof

We are required to prove

$$P\left(\mu \in \left(-\infty, \overline{X} + t_{\alpha,n-1}\frac{S}{\sqrt{n}}\right)\right) = 1-\alpha.$$

Proof (Cont.)

We have

$$\begin{aligned}\text{LHS} &= P\left(\mu < \overline{X} + t_{\alpha,n-1}\frac{S}{\sqrt{n}}\right) = P\left(-t_{\alpha,n-1}\frac{S}{\sqrt{n}} < \overline{X} - \mu\right)\\ &= P\left(-t_{\alpha,n-1} < (\overline{X} - \mu)/\frac{S}{\sqrt{n}}\right)\\ &= P(-t_{\alpha,n-1} < T)\\ &= 1 - \alpha\end{aligned}$$

To prove the last equality draw a picture - I want *you* to draw the picture on tests and the homework. □

Once we have an actual sample $x_1, x_2, \dots, x_n$ we obtain the observed value $\overline{x}$ for the random variable $\overline{X}$ the observed value s for the random variable S and the observed value $\left(-\infty, \overline{x} + t_{\alpha,n-1}\frac{s}{\sqrt{n}}\right)$ for the confidence (random) interval $\left(-\infty, \overline{X} + t_{\alpha,n-1}\frac{S}{\sqrt{n}}\right)$. The observed value of the confidence (random) interval is also called the lower-tailed $100(1-\alpha)\%$ confidence interval for μ.

The random variable $\overline{X} + t_{\alpha,n-1}\dfrac{S}{\sqrt{n}}$ or its observed value the number $\overline{x} + t_{\alpha,n-1}\dfrac{s}{\sqrt{n}}$ is often called a confidence *upper bound* for μ because

$$P\left(\mu < \overline{X} + t_{\alpha,n-1}\frac{S}{\sqrt{n}}\right) = 1 - \alpha.$$

4. The upper-tailed confidence interval for μ

Problem Prove the following theorem.

Theorem 4

The random interval $\left(\overline{X} - t_{\alpha,n-1}\dfrac{S}{\sqrt{n}}, \infty\right)$, *is a* $100(1-\alpha)\%$ *confidence interval for* μ.

The random variable $\overline{X} - t_{\alpha,n-1}\dfrac{S}{\sqrt{n}}$ or its observed value the number $\overline{x} - t_{\alpha,n-1}\dfrac{s}{\sqrt{n}}$ is often called a confidence *lower bound* for μ because

$$P\left(\mu > \overline{X} - t_{\alpha,n-1}\frac{S}{\sqrt{n}}\right) = 1 - \alpha.$$