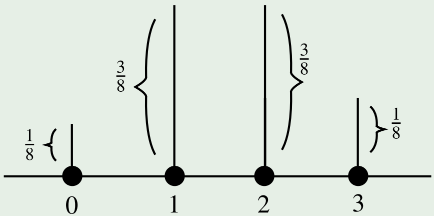
Lecture 9 : Change of discrete random variable

You have already seen (I hope) that whenever you have "variables" you need to consider *change of variables*. Random variables are no different.
The notion of "change of random variable" is handled too briefly on page 112 and 115 (the meaning of the symbol $h(X)$ is not even defined in the text). *This is something I will test you on.*

Example 1

Suppose $X \sim Bin\left(3, \frac{1}{2}\right)$.

line graph



table

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(b)

Suppose we want to define a new random variable $Y = 2X - 1$.
How do we do it?
So how do we define $P(Y = k)$?
Answer - express Y in terms of X and compute so

$$\begin{aligned} P(Y = k) &= P(2X - 1 = k) \\ &= P\left(X = \frac{k+1}{2}\right) \qquad (*) \end{aligned}$$

The right-hand site is the logical *definition* of the left-hand side.
But as is often the case in probability it is easier to pretend we know what $P(Y = k)$ means already and then the last two steps are a computation.

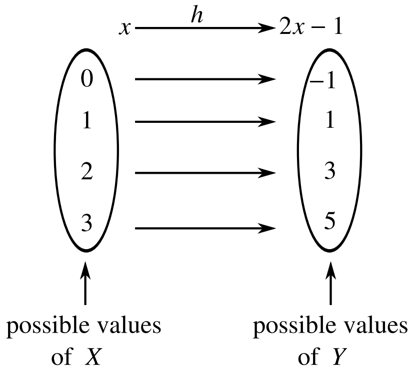
So let's compute the *pmf* of Y.
What are the possible values of Y?
From (*) k is a possible value of $Y \Leftrightarrow \dfrac{k+1}{2}$ is a possible values of X.

$$\Longleftrightarrow \quad \frac{k+1}{2} = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \quad \Longleftrightarrow \quad Y = \begin{cases} -1 \\ 1 \\ 3 \\ 5 \end{cases}$$

Note

$$\frac{k+1}{2} = \underbrace{x}_{\text{possible value of } X} \Leftrightarrow \underbrace{k}_{\text{possible value of } Y} = 2x - 1$$

So the possible values of Y are obtained by applying the function $h(x) = 2x - 1$ to the possible values of X.
(note $Y = f(X)$).



Just "push forward" the values of X.

Now we have computed the possible values of Y we need to compute their probabilities. Just repeat what we did

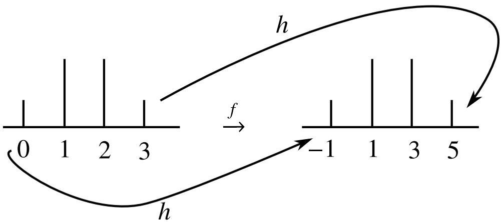
$$\begin{aligned} P(Y=-1) &= P(2X-1=-1) \\ &= P(X=0) = \frac{1}{8} \\ P(Y=1) &= P(2X-1=1) \\ &= P(X=1) = \frac{3}{8} \end{aligned}$$

Similarly

$$P(Y=3) = \frac{3}{8} \quad \text{and} \quad P(Y=5) = \frac{1}{8}$$

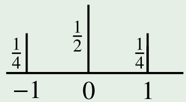
y	-1	1	3	5
$P(Y=y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

So we have the “same probabilities” as before namely $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ it is just then are pushed-forward to new locations



Example 2 (Probabilities can "coalesce")

There is one tricky point. Several different possible values of X can push-forward to the same values of Y. We now give an example.
Suppose X has pmf



That is

$$P(X=-1)=\frac{1}{4},\quad P(X=0)=\frac{1}{2},\quad P(X=1)=\frac{1}{4}$$

We will make the change of variable $Y = X^2$. So what happens when we push forward the three values -1, 0, 1 by $h(x) = x^2$.
We get only the *two* values 0 and 1.

$$\begin{array}{ccc} -1 & \xrightarrow{h(x)} & 1 \\ 0 & \longrightarrow & 0 \\ 1 & \longrightarrow & 1 \end{array}$$

What happens with the corresponding probabilities

$$P(Y = 0) = P(X^2 = 0) = P(X = 0) = \frac{1}{2}$$

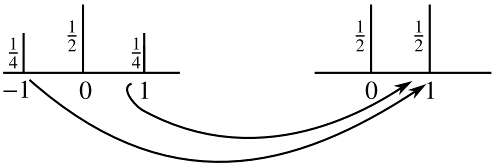
But

$$\begin{aligned} P(Y = 1) = P(X^2 = 1) &= P(X = 1 \ \text{ or } \ X = -1) \\ &= P((X = 1) \cup (X - 1)) \\ &= P(X = 1) + P(X = -1) \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{aligned}$$

So we set

y	0	1
$P(Y=y)$	$\frac{1}{2}$	$\frac{1}{2}$

So,



Think of two masses (probabilities) of mass $\dfrac{1}{4}$, one at -1, and one at !
coalescing into a combined mass of $\dfrac{1}{2}$ at 0.

The Expected Value Formula

If $h(x)$ in the transformation law $Y = h(X)$ is complicated it can be very hard to explicitly compute the *pmf* of Y. Amazingly we can compute the expected value $E(Y)$ using the old proof $p_X(x)$ of X according to

Theorem 3

$$E(h(X)) = \sum_{\substack{\textit{possible} \\ \textit{values of } X}} h(x)p_X(x) \qquad = \sum_{\substack{\textit{possible values} \\ \textit{of } X}} h(x)P(X = x)$$

We will illustrate this with the *pmf*'s of Example 1.

First we compute $E(Y)$ *using the definition of* $E(Y)$.

$$\begin{array}{c|c|c|c|c|} y & -1 & 1 & 3 & 5 \\ \hline P(Y=y) & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ \hline \end{array} \qquad (\sharp)$$

$$\begin{aligned} E(Y) &= \sum_{\substack{\text{possible value} \\ \text{of } Y}} y\, P(Y=y) \\ &= (-1)\left(\frac{1}{8}\right) + (1)\left(\frac{3}{8}\right) + (3)\left(\frac{3}{8}\right) + (5)\left(\frac{1}{8}\right) \\ &= \frac{-1+3+9+5}{8} \\ &= \frac{16}{8} = 2 \end{aligned}$$

Notice to do the previous computations we needed the table ($\sharp$) which we computed five pages ago.
Now we use the *Theorem*.
So now we use that Y is a function of the random variable X and use the proof of X from the table on page 1.

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(b)

$$\begin{aligned} E(X) &= \sum_{\substack{\text{possible values} \\ \text{of } X}} h(x)P(X=x) \\ &= \sum_{x=0,1,2,3} (2x-1)P(X=x) \\ &= (-1)\left(\frac{1}{8}\right) + (1)\left(\frac{3}{8}\right) + (3)\left(\frac{5}{8}\right) + (5)\left(\frac{3}{8}\right) = 2 \end{aligned}$$

The most common change of variable is linear $Y = aX + b$ so we will give formulas to show how expected value and variance behave under such a change.

Theorem

(i) $E(aX + b) = aE(X) + b$

(ii) $V(aX + b) = a^2 V(X)$

(so $V(-X) = V(X)$)