The t tests

March 14, 2006

1 Introduction

In this lecture we will derive the formulas for the two-sided t-test and the upper-tailed t-test for the mean in a normal distribution *when the variance σ^2 is unknown*. Let $x_1, x_2, \cdots, x_n$ be a sample from a normal distribution with mean μ and variance σ^2. Recall that $\overline{X}$ is the sample mean (the point estimator for the populations mean μ) .

2 The two-sided t-test

We wish to give a test to decide between:

$$H_0 : \mu = \mu_0$$
$$H_a : \mu \neq \mu_0$$

The two-sided t-test is the decision rule:
reject H_0 if either $\bar{x} \leq \mu_0 - t_{\alpha/2,n-1}(\frac{s}{\sqrt{n}})$ or $\bar{x} \geq \mu_0 + t_{\alpha/2,n-1}(\frac{s}{\sqrt{n}})$.
We will now prove that the two-sided t-test has significance level (i.e. Type I error probability) equal to α. We will need the following theorem from Probability Theory.

Theorem 1. $T = (\overline{X} - \mu)/(\frac{S}{\sqrt{n}})$ *has standard normal distribution.*

We now prove

Theorem 2. *The two-sided t-test has significance level α.*

Proof.

$$P(\text{Type I error}) = P(\text{Reject } H_0 \text{ when } H_0 \text{ is correct})$$

$$= P(\bar{X} \leq \mu_0 - t_{\alpha/2,n-1}(\frac{S}{\sqrt{n}}) \text{ or } \bar{X} \geq \mu_0 + t_{\alpha/2,n-1}(\frac{S}{\sqrt{n}}) \text{ when } \mu = \mu_0)$$

$$=P(\bar{X} - \mu_0 \leq -t_{\alpha/2,n-1}(\frac{S}{\sqrt{n}}) \text{ or } \bar{X} - \mu_0 \geq t_{\alpha/2,n-1}(\frac{S}{\sqrt{n}}) \text{ when } \mu = \mu_0)$$

$$=P((\overline{X} - \mu_0)/(\frac{S}{\sqrt{n}}) \leq -t_{\alpha/2,n-1} \text{ or } (\overline{X} - \mu_0)/(\frac{S}{\sqrt{n}}) \geq t_{\alpha/2,n-1} \text{ when } \mu = \mu_0).$$

Now we use the assumption that $\mu = \mu_0$ to replace μ_0 by μ in the ratio $(\overline{X} - \mu_0)/(\frac{S}{\sqrt{n}})$. Then we apply Theorem 1 above to deduce that the rewritten ratio $T = (\overline{X} - \mu)/(\frac{S}{\sqrt{n}})$ has t-distribution with $n-1$ degrees of freedom. Thus we obtain the new equation

$$P(\text{Type I error}) = P((T \leq -t_{\alpha/2,n-1} \text{ or } T \geq t_{\alpha/2,n-1}) = P((T \leq -t_{\alpha/2,n-1}) + P(T \geq t_{\alpha/2,n-1}).$$

Each of the two probabilities in the last term are equal to $\alpha/2$. To prove this draw a picture, the second is equal to $\alpha/2$ by definition, the second by symmetry.

□

3 The upper-tailed t-test

We wish to give a test to decide between:

$$H_0 : \mu = \mu_0$$
$$H_a : \mu > \mu_0$$

The upper-tailed t-test is the decision rule:
reject H_0 if $\bar{x} \geq \mu_0 + t_{\alpha,n-1}(\frac{s}{\sqrt{n}})$.
We will now prove that the two-sided t-test has significance level (i.e. Type I error probability) equal to α. Once again we will need Theorem 1.
We now prove

Theorem 3. *The upper-tailed t-test has significance level α.*

Proof.

$$P(\text{Type I error}) = P(\text{Reject } H_0 \text{ when } H_0 \text{ is correct})$$
$$= P(\bar{x} \geq \mu_0 + t_{\alpha,n-1}(\frac{S}{\sqrt{n}}) \text{ when } \mu = \mu_0)$$
$$P(\bar{X} - \mu_0 \geq t_{\alpha,n-1}(\frac{S}{\sqrt{n}}) \text{ when } \mu = \mu_0)$$
$$=P((\overline{X} - \mu_0)/(\frac{S}{\sqrt{n}}) \geq t_{\alpha,n-1} \text{ when } \mu = \mu_0).$$

Now we use the assumption that $\mu = \mu_0$ to replace μ_0 by μ in the ratio $(\overline{X} - \mu_0)/(\frac{s}{\sqrt{n}})$. Then we apply Theorem 1 above to deduce that the rewritten ratio $T = (\overline{X} - \mu)/(\frac{S}{\sqrt{n}})$ has t-distribution with $n-1$ degrees of freedom. Thus we obtain the new equation

$$P(\text{Type I error}) = P(T \geq t_{\alpha,n-1}).$$

This last probability is equal to α by definition (draw a picture). $\square$