The p-values of the z-tests

March 14, 2006

1 Introduction

In this lecture we will derive the formulas for the p-values of the two-sided z-test and the upper-tailed z-test. Read the proof for the upper-tailed z-test because it is simpler (the two-sided test involves one more trick, introducing the absolute value of z).

We recall that the p-value of a test (decision rule) for a given sample is the smallest value of α for which H_0 will be rejected using the given sample.

2 The p-value of the two-sided z-test

Let $x_1, x_2, \cdots, x_n$ be a sample from a normal distribution with unknown mean μ and known variance σ^2. We wish to decide between:

$$H_0 : \mu = \mu_0$$
$$H_a : \mu \neq \mu_0$$

The two-sided z-test is the decision rule:

$$\text{reject } H_0 \text{ if either } \bar{x} \leq \mu_0 - z_{\alpha/2}(\frac{\sigma}{\sqrt{n}}) \text{ or } \bar{x} \geq \mu_0 + z_{\alpha/2}(\frac{\sigma}{\sqrt{n}}).$$

We compress this decision rule by putting

$$z = (\bar{x} - \mu_0)/(\frac{\sigma}{\sqrt{n}}).$$

Note that z is a function of $\bar{x}$ and hence is function of the sample $x_1, x_2, \cdots, x_n$. The compressed decision rule (equivalent to the one above) is then: reject H_0 if

$$\text{either } z \leq -z_{\alpha/2} \text{ or } z \geq z_{\alpha/2}. \tag{1}$$

We can compress this decision rule still more by introducing the absolute value $|z|$ and noting the previous two inequalities in z can be combined into one inequality in $|z|$. The above rejection rule is equivalent to: reject H_0 if

$$|z| \geq z_{\alpha/2}. \tag{2}$$

We are now ready to prove the formula for the p-value for the two-sided z-test. Note that the data has been coded into z.

Theorem 1. *The p-value of the two-sided z-test is a function of z alone and moreover*

$$p = p(z) = 2(1 - \Phi(|z|).$$

Proof. The set of α's for which H_0 will be rejected is the set of α's that satisfy the previous *nonlinear* inequality (2) in α. The trick to compute p-value is to apply the standard normal cdf Φ to both sides of the inequality (2). Since Φ is an increasing function we obtain

$$\Phi(|z|) \geq \Phi(z_{\alpha/2}).$$

But we have (draw a picture)

$$\Phi(z_{\alpha/2}) = 1 - \alpha/2$$

and we obtain the following em linear inequality in α which is equivalent to the inequality (2) - all the steps we made were reversible.

$$\Phi(|z|) \geq 1 - \alpha/2 \Leftrightarrow \alpha/2 \geq 1 - \Phi(|z|) \Leftrightarrow \alpha \geq 2(1 - \Phi(|z|).$$

Thus the set of α's for which H_0 will be rejected is the set of α's satisfying the linear inequality

$$\alpha \geq 2(1 - \Phi(|z|).$$

The smallest such α is obviously $2(1 - \Phi(|z|)$. □

3 The p-value of the upper-tailed z-test

Let $x_1, x_2, \cdots, x_n$ be a sample from a normal distribution with unknown mean μ and known variance σ^2. We wish to decide between:

$$H_0 : \mu = \mu_0$$
$$H_a : \mu > \mu_0$$

The upper-tailed z-test is the decision rule:

$$\text{reject } H_0 \text{ if } \bar{x} \geq \mu_0 + z_\alpha(\frac{\sigma}{\sqrt{n}}).$$

We compress this decision rule by putting

$$z = (\bar{x} - \mu_0)/(\frac{\sigma}{\sqrt{n}}).$$

Note that z is a function of $\bar{x}$ and hence is function of the sample $x_1, x_2, \cdots, x_n$. The compressed decision rule (equivalent to the one above) is then:
reject H_0 if

$$z \geq z_\alpha. \tag{3}$$

We are now ready to prove the formula for the p-value for the upper-tailed z-test. We don't need the absolute value $|z|$ for the one-sided tests. Note that the data has been coded into z.

Theorem 2. *The p-value of the upper-tailed z-test is a function of z alone and moreover*

$$p = p(z) = 1 - \Phi(z).$$

Proof. The set of α's for which H_0 will be rejected is the set of α's that satisfy the previous *nonlinear* inequality (3) in α. The trick to compute p-value is to apply the standard normal cdf Φ to both sides of the inequality (3). Since Φ is an increasing function we obtain

$$\Phi(z) \geq \Phi(z_\alpha).$$

But we have (draw a picture)

$$\Phi(z_\alpha) = 1 - \alpha$$

and we obtain the following *linear* inequality in α which is equivalent to the inequality (3) - all the steps we made were reversible.

$$\Phi(z) \geq 1 - \alpha \Leftrightarrow \alpha \geq 1 - \Phi(z).$$

Thus the set of α's for which H_0 will be rejected is the set of α's satisfying the linear inequality

$$\alpha \geq 1 - \Phi(z).$$

The smallest such α is obviously $1 - \Phi(z)$. $\square$