The z tests

March 13, 2006

1 Introduction

In this lecture we will derive the formulas for the two-sided z-test and the upper-tailed z-test for the mean in a normal distribution *when the variance σ^2 is known.* Let $x_1, x_2, \cdots, x_n$ be a sample from a normal distribution with mean μ and variance σ^2. Recall that $\overline{X}$ is the sample mean (the point estimator for the populations mean μ) .

2 The two-sided z-test

We wish to give a test to decide between:

$$H_0 : \mu = \mu_0$$
$$H_a : \mu \neq \mu_0$$

The two-sided z-test is the decision rule:
reject H_0 if either $\bar{x} \leq \mu_0 - z_{\alpha/2}(\frac{\sigma}{\sqrt{n}})$ or $\bar{x} \geq \mu_0 + z_{\alpha/2}(\frac{\sigma}{\sqrt{n}})$.
We will now prove that the two-sided z-test has significance level (i.e. Type I error probability) equal to α. We will need the following theorem from Probability Theory.

Theorem 1. *$Z = (\overline{X} - \mu)/(\frac{\sigma}{\sqrt{n}})$ has standard normal distribution.*

We now prove

Theorem 2. *The two-sided z-test has significance level α.*

Proof.

$$
\begin{aligned}
&P(\text{Type I error}) = P(\text{Reject } H_0 \text{ when } H_0 \text{ is correct})\\
&= P(\bar{X} \leq \mu_0 - z_{\alpha/2}(\frac{\sigma}{\sqrt{n}}) \text{ or } \bar{X} \geq \mu_0 + z_{\alpha/2}(\frac{\sigma}{\sqrt{n}}) \text{ when } \mu = \mu_0)\\
&=P(\bar{X} - \mu_0 \leq -z_{\alpha/2}(\frac{\sigma}{\sqrt{n}}) \text{ or } \bar{X} - \mu_0 \geq z_{\alpha/2}(\frac{\sigma}{\sqrt{n}}) \text{ when } \mu = \mu_0)\\
&=P((\overline{X} - \mu_0)/(\frac{\sigma}{\sqrt{n}}) \leq -z_{\alpha/2} \text{ or } (\overline{X} - \mu_0)/(\frac{\sigma}{\sqrt{n}}) \geq z_{\alpha/2} \text{ when } \mu = \mu_0).
\end{aligned}
$$

Now we use the assumption that $\mu = \mu_0$ to replace μ_0 by μ in the ratio $(\overline{X} - \mu_0)/(\frac{\sigma}{\sqrt{n}})$. Then we apply Theorem 1 above to deduce that the rewritten ratio $Z = (\overline{X} - \mu)/(\frac{\sigma}{\sqrt{n}})$ has standard normal distribution. Thus we obtain the new equation

$$P(\text{Type I error}) = P((Z \leq -z_{\alpha/2} \text{ or } Z \geq z_{\alpha/2}) = P((Z \leq -z_{\alpha/2}) + P(Z \geq z_{\alpha/2}).$$

Each of the two probabilities in the last term are equal to $\alpha/2$. To prove this draw a picture, the second is equal to $\alpha/2$ by definition, the second by symmetry.

□

3 The upper-tailed z-test

We wish to give a test to decide between:

$$
\begin{aligned}
H_0 &: \mu = \mu_0\\
H_a &: \mu > \mu_0
\end{aligned}
$$

The upper-tailed z-test is the decision rule:
reject H_0 if $\bar{x} \geq \mu_0 + z_\alpha(\frac{\sigma}{\sqrt{n}})$.
We will now prove that the two-sided z-test has significance level (i.e. Type I error probability) equal to α. Once again we will need Theorem 1.
We now prove

Theorem 3. *The upper-tailed z-test has significance level α.*

Proof.

$$P(\text{Type I error}) = P(\text{Reject } H_0 \text{ when } H_0 \text{ is correct})$$
$$= P(\bar{x} \geq \mu_0 + z_\alpha(\frac{\sigma}{\sqrt{n}}) \text{ when } \mu = \mu_0)$$
$$P(\bar{X} - \mu_0 \geq z_\alpha(\frac{\sigma}{\sqrt{n}}) \text{ when } \mu = \mu_0)$$
$$=P((\overline{X} - \mu_0)/(\frac{\sigma}{\sqrt{n}}) \geq z_\alpha \text{ when } \mu = \mu_0).$$

Now we use the assumption that $\mu = \mu_0$ to replace μ_0 by μ in the ratio $(\overline{X} - \mu_0)/(\frac{\sigma}{\sqrt{n}})$. Then we apply Theorem 1 above to deduce that the rewritten ratio $Z = (\overline{X} - \mu)/(\frac{\sigma}{\sqrt{n}})$ has standard normal distribution. Thus we obtain the new equation

$$P(\text{Type I error}) = P(Z \geq z_\alpha).$$

This last probability is equal to α by definition (draw a picture). □