LECTURE 2

Counting Techniques §2.3

The Three Basic Rules

1. The Product Rule for Ordered Pairs and Ordered k-tuples.

Our first counting rule applies to any situation in which a set consists of *ordered* pairs of objects (a, b) where a comes from a set A and b comes from a set B. In terms of pure mathematics the *Cartesian product* $A \times B$ is the set of such pairs

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Proposition (text pg. 60)
If the first element of the ordered pair can be selected in n_1 ways and *if for each of these* n_1 *ways* the second element can be selected in n_2 ways then the number of pairs is $n_1 n_2$.
Mathematically, if

$$\#(A) = n_1 \text{ and } \#(B) = n_2 \text{ then } \#(A \times B) = n_1 n_2.$$

There are analogous results for ordered triples, etc.

$$\#(A \times B \times C) = n_1\ n_2\ n_3.$$

Example
How many "words" of two letters can we make from the alphabet of five letters $\{a, b, c, d, e\}$.

Solution (Note that order counts $ab \neq ba$).
There are two ways to think about the problem pictorially.

1. Filling in two slots $- -$

We have a choice of 5 ways to fill in the first slot and for each of these we have 5 more ways to fill in the second slot so we have 25 ways

$$\underline{5}\ \underline{5} = 25.$$

2. Draw a tree where each edge is a choice

The number of pairs is the number paths from the root to a "leaf" (i.e., a node at the far right).

In this case there are 25 paths.

Problem
How many words are there of length 3?.

There is another mathematical interpretation of the product rule which we will use in the "birthday problem". Let S be a set with ℓ elements. Suppose we wish to count the number of ways to fill k slots $- - - \cdot - - -$ as above using elements of S. So we wish to count all words $\underline{s}_1, \underline{s}_2, \cdots, \underline{s}_k$ using the elements in S as alphabet. Now associate to the above word w the mapping f_w from then k element set $\{1, 2, \cdots, k\}$ to the set S given by

$$f(1) = s_2, f(2) = s_2, \cdots f(k) = s_k.$$

So we see that the number of words as above (which we know is $k\ell$ by the product rule) is the same as the number of mappings from then k element set $\{1, 2, \cdots, k\}$ to the ℓ set S and in fact we have
Proposition
The number of mappings from a k-element set to an ℓ-element set if $k\ell$.

2. Permutations (pg. 62)

In the previous problem the word $a\ a$ was allowed. What if we required the letters in the word to be distinct. Then we would get 2-permutations from the 5-element set $\{a, b, c, d\}$ according to the following definition.

Definition
An *ordered* sequence of k *distinct* objects taken from a set of n elements is called a *k-permutation* of the n objects. The number of k-permutations of the n objects will be denoted $P_{k,n}$. So *order counts.*

Let us return to our 5-element set $\{a, b, c, d, e\}$ and count the number of 2-permutations. It is best to think in terms of slots

$$\underline{5}\ \underline{4} = 20.$$

There are 5 choices for the first slot but only 4 for the second because whatever we put in the first slot connect be put in the second slot so $P_{2,5} = 20$.

What is $P_{3,5}$?

Proposition (text pg. 68)

$$P_{k,n} = \underbrace{n(n-1)(n-2) - (n-k+1)}_{k \text{ terms}}.$$

Proof
Fill in k slots with no repetitions

$$\underbrace{\underline{n\quad}\quad \underline{n-1}\quad \underset{\text{how many}}{\underline{\quad\downarrow\quad}}\quad \underline{\qquad}\quad \underset{\text{how many}}{\underline{\quad\downarrow\quad}}}_{k}.$$

$\square$

Note that if we allowed repetitions we would get n^k.

$$\underbrace{\underline{n\quad}\ \ \underline{n\quad}\ \ \underline{n\quad}\ \ \underline{\quad\ }\ \ \underline{n\quad}}_{k}.$$

There is a very important special case

$$P_{n,n} = n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1.$$

There are $n!$ ways to take n distinct objects and arrange them in order.

Example

$n = 3, \{a, b, c\}$.

$$\left.\begin{array}{l} abc \\ acb \\ \\ bac \\ bca \\ \\ cab \\ cba \end{array}\right\} 3! = (3)(2)(1) = 6.$$

When you list objects it is helpful to list them in dictionary order.

A Better Formula for $P_{k,n}$.

Here is a better formula for $P_{k,n}$.

Proposition

$$P_{k,n} = \frac{n!}{(n-k)!}.$$

Proof This is an algebraic trick

$$\frac{n!}{(n-k)!} = \frac{n(n1)\cdots(n-k+1)\overbrace{(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1}^{(n-k)!}}{(n-k)!}.$$

So cancel the second part of the numerator with the denominator

$$\frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1) = P_{k,n}.$$

$\square$

Just as there was another mathematical interpretation of the product rule there is an interpretation of k permutations of an n element set S in terms of mappings. Let S be a set with n elements. Suppose we wish to count the number of ways to fill k slots $---\cdot---$ as above using elements of S *such that all the elements we use are different*. So we wish to count all words $\underline{s}_1, \underline{s}_2, \cdots, \underline{s}_k$ using the elements in S as alphabet such that $s_i \neq s_j$ for $i \neq j$. Now as before associate to the above word w the mapping f_w from then k element set $\{1, 2, \cdots, k\}$ to the set S given by

$$f(1) = s_2, f(2) = s_2, \cdots f(k) = s_k.$$

So how does f_w reflect that fact that all the s_i's are distinct? Answer: the s_i's are distinct if and only if the mapping f_w is $1:1$.

So we see that the number of words as above (which we know is $k\ell$ by the product rule) is the same as the number of mappings from then k element set $\{1, 2, \cdots, k\}$ to the ℓ set S and in fact we have

Proposition

The number of $1:1$ mappings from a k-element set to an n-element set is $\frac{n!}{(n-k)!}$.

The Birthday Problem

Suppose there are n people in a room. What is the probability B_n that at least two people have the same birthday (e.g., March 11)? It is customary to ignore leap years for this problem.

Let S be the set of all possible birthdays for the n people. What is the cardinality of S? We will use the product rule. For each person in the room create a slot so we get

$$\underbrace{\_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_}_{n \text{ people}}$$

Now fill in each slot with a day of the year (assign a birthday to each person). How can we do this? We have 365 ways to fill in the first slot, 365 ways to fill in the second slot and finally 365 ways to fill in the n-th and last slot. The picture is

$$\underbrace{\underline{365}\ \underline{365}\cdots\underline{365}}_{n \text{ people}}$$

Hence

$$\sharp(S) = (365)^n.$$

Now let $A \subset \mathcal{S}$ be the event that at least two people have the same birthday. So the complement A' is So

$$A' = \text{all the people in the room have different birthdays.}$$

So

$$B_n = 1 - P(A').$$

Now what is A'? Again we create a slot for each person but now we have to put 364 into the second slot, 363 into the third etc. (in order that the birthdays be distinct). So we have the filling

$$\underbrace{\underline{365}\ \underline{365} \cdots \underline{365}}_{n \text{ people}}$$

Hence

$$\sharp(S) = (365)^n.$$

and

$$\sharp(A') = P_{n,365} = \frac{365!}{(365-n)!}$$

and

$$P(A') = \frac{\left(\frac{365!}{(365-n)!}\right)}{(365)^n} = \frac{365!}{(365-n)!(365)^n}.$$

Hence we have

$$B_n = 1 - \frac{365!}{(365-n)!(365)^n}.$$

I will now give a table of B_n which I got from Wikipedia (by searching on "the birthday problem").

n	B_n
10	.117
20	. 411
23	.507
30	.706
50	.970
57	.990
10	.9999997

3. Combinations

There are many counting problems in which one is given a set of n objects and one wants to count the number of *unordered* subsets with k elements.

An unordered subset with k-elements taken from a set of n elements is called a k-combination of that set. The number of k-combinations is denoted $C_{k,n}$.

Which is bigger $C_{k,n}$ or $P_{k,n}$?

What is $C_{n,n}$?

Example

$$P_{2,3} = 6 \supset C_{2,3} = 3$$
$$S = \{a, b, c, \}$$

2 permutations of S	2 combinations of S
$ab\ ba$	$\{a, b\}$
$bc\ cb$	$\{b, c\}$
$ac\ ca$	$\{a, c\}$

Each two combinations gives rise to 2 2-permutations. So

$$P_{2,3} = 2C_{2,3} = (2)(3) = 6.$$

The Formula for $C_{k,n}$

Proposition (pg. 64)

$$P_{k,n} = C_{k,n} \cdot k!$$

so

$$C_{k,n} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

Proof

To make a k-permutation first make an unordered choice of the k-elements, i.e., choose a k-combination, then, for each such choice arrange the elements in order (there are $P_{k,k} = k!$ ways to do this). So we have

$$\#(k\text{-permutations}) = \#(k\text{-combinations}) \cdot k!$$

$\square$

More Notation

The binomial coefficient $\begin{pmatrix} n \\ k \end{pmatrix}$ is defined by

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}.$$

This is because

$$\underbrace{(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k b^{n-k}.}_{\text{The binomial theorem}}$$

So

$$C_{k,n} = \left(\begin{array}{c} n \\ k \end{array} \right).$$

We will use $\left(\begin{array}{c} n \\ k \end{array} \right)$ instead of $C_{k,n}$.

The Toast Problem

When my wife and I were on a trip to Spain with our church we had 20 people at dinner. We all clinked (is this a genuine English word) our classes. I dazzled my friends by telling how many clinks there were.

Now you can answer this question – how many?

More Problems

1. How many 5 card poker hands are there?
2. How many 13 card bridge hands are there?

Lastly,

Proposition

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \left(\begin{array}{c} n \\ n-k \end{array} \right).$$

Proof (Challenge):
Find two proofs, one "combinatorial" and one algebraic.