LECTURE 5

Independence §2.5

Definition. *Two events A and B are independent if*

$$P(A|B) = P(A) \tag{$\sharp$}$$

otherwise they are said to be dependent.

The equation $P(A|B) = P(A)$ says that the knowledge that B has occurred does not effect the probability A will occur.

(#) appears to be asymmetric but we have (assuming $P(A) \neq 0$ so $P(B|A)$ is defined and $P(B) \neq 0$ so $P(A|B)$ is defineD).

Proposition.

$$P(A|B) = P(A) \Longleftrightarrow P(B|A) = P(B)$$

Proof

$$\begin{aligned} P(A \cap B) &= P(A)P(B|A) \\ P(B \cap A) &= P(B)P(A|B). \end{aligned}$$

But $A \cap B = B \cap A$ (this is the point), so

$$P(A)P(B|A) = P(B)P(A|B)$$

so

$$\frac{P(B|A)}{P(B)} = \frac{P(A|B)}{P(A)}$$

Then

$$LHS = 1 \Longleftrightarrow RHS = 1.$$

□

The Standard Mistake

The English language can trip us up here. Suppose A and B are <u>mutually exclusive</u> events ($A \cap B = \phi$) with $P(A) \neq 0$ and $P(B) \neq 0$. Are A and B independent?

$$P(A|B)\ \frac{P(A \cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = \frac{0}{P(B)} = 0$$

so $P(A|B) \neq P(A)$.

In this case if you know B has occurred <u>then A cannot occur</u> at all. This is the <u>opposite</u> of independence.

Two Contrasting Examples

1. Our favorite example

$$A = \heartsuit \text{ on } 1^{st}$$
$$B = \heartsuit \text{ on } 2^{nd}$$

$$P\left(\heartsuit \text{ on } 2^{nd} | \heartsuit \text{ on } 1^{st}\right) = \frac{12}{51} \quad (*)$$

$P\left(\heartsuit \text{ on } 2^{nd} \text{ with no other information}\right) = \frac{13}{52}$, so $P(B|A) \neq P(B)$, so A and B are not independent.

2. Our very first example Flip a fair coin twice

$$A = H \text{ on } 1^{st}$$
$$B = H \text{ on } 2^{nd}$$

$$P\left(H \text{ on } 2^{nd} | H \text{ on } 1^{st}\right) = \frac{1}{2} \quad (**)$$

$P\left(H \text{ on } 2^{nd}\right) = \frac{1}{2}$, so $P(B|A) = P(A)$, so A and B are independent. Hence,

$$P(A \cap B) = P(A)P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

as we saw in Lecture 1.

Independence of more than two elements

Definition. *The events $A_1, A_2, \cdots, A_n$ are independent if for every k and for every collection of k distinct indices $i_1, i_2, \cdots, i_k$ drawn from $1, 2, \cdots, n$ we have*

$$(\flat) \qquad (A_{i_1}, \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}).$$

So in particular ($k = n$) we have

$$(\#) \qquad P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2) \cdots P(A_n)$$

however, there are examples where (#) holds but ($\flat$) fails for some $k < n$ so the events are not independent.

Example $n = 3$

Special case of the definition
Three events A, B, C are indpendent if

$$(\#) \qquad P(A \cap B \cap C) = P(A)P(B)P(C)$$

and

$(\flat_1)$ $$P(A \cap B) = P(A)P(B)$$

$(\flat_2)$ $$P(A \cap C) = P(A)P(C)$$

$(\flat_3)$ $$P(B \cap C) = P(B)P(C)$$

To specialize what I said before there are examples where (#) holds but one of the ($\flat$)'s fails so (#) alone does not imply independence.

System/Component Problems I will now do one of my favorite types of problems (they often turn up on midterms and finals). See Example 2.35, pg.79 and Problems 80 and 87 pg.81. Consider the following system S. Suppose each of th three components has probability p of working. Suppose all components function independently. What is the probability the system will work, ie., an input signal on the left will come out on the right.

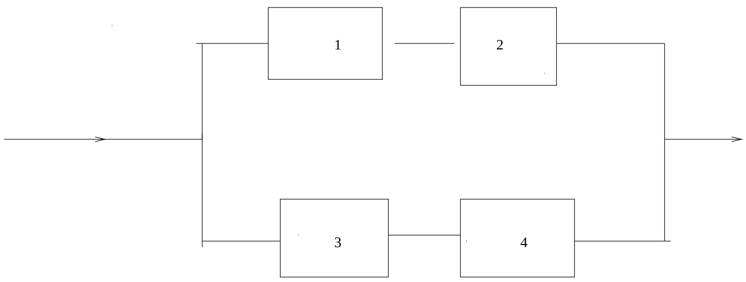

Figure 1: A system with four components.

Solution
It is important that you follow the format below – don't skip steps. Skipping steps is fatal in mathematics (as in almost everything).

Define events

$$S = \text{system works}$$

$$A_i = \text{i-th component works} \quad i = 1, 2, 3, 4$$

The hard part.) Express the set S in terms of the sets A_1, A_2, A_3 using the geometry of the system. Define two subsystems **top** and **bottom** by taking **top** to be the top wire and the top two components and **bottom** to be the bottom wire and the bottom two components. Define two new events T and B by

$$T = \text{top works}$$
$$B = \text{bottom works}$$

The first key observation is

$$S = T \cup B$$

and hence

$$P(S) = P(T) + P(B) - P(T \cap B).$$

Now we need the "grouping principle". If you take a collection of independent events and split them into two groups it with no event in common then the events you get by forming any word form the first group (using unions and intersections) will be independent of any word you get from the second group. Hence it T and B are independent- take as the first group $\{A_1, A_2\}$ and take as the second group $\{A_3, A_4\}$. Hence

$$P(T \cap B) = P(T)P(B).$$

So it remains to compute $P(T)$ and $P(B)$.

But

$$T = A_1 \cap A_2$$

and

$$B = A_3 \cap A_4$$

so

$$P(T) = P(A_1 \cap A_2) = P(A_1)P(A_2) = p^2.$$

Similarly

$$P(B) = p^2.$$

Hence we obtain

$$P(S) = p^2 + p^2 - p^4 = 2p^2 - p^4.$$