# LECTURE 8

## 1 The General Distribution

The geometric distribution is a special case of negative binomial, it is the case $r = 1$. It is so important we give it special treatment.

**Motivating example**
Suppose a couple decide to have children until they have a girl. Suppose the probability of having a girl is $p$. Let

$$X = \text{the number of boys that precede the first girl.}$$

Find the probability distribution of $X$. First $X$ could have any possible whole number value (although $X = 1,000,000$ is very unlikely).

$$P(X = k) = P\left(\underbrace{\underline{B}\ \underline{B}\ \underline{B}\ \_\ \_\ \underline{B}\ \underline{G}}_{k}\right).$$

$$= q^k p \quad (\text{where} \quad q = 1 - p).$$

We have, suppose births are independent. We have motivated

**Definition.** *Suppose a discrete random variable $X$ has the following pmf*

$$P(X = k') = q^k p, 0 \leq k < \infty.$$

*The $X$ is said to have geometric distribution with parameter $p$.*

**Remark .** *Usually, this is developed by replacing "having a child" by a Bernoulli experiment and "having a girl" by a "success" (PC). I could have used coin flips.*

**Proposition.** *Suppose $X$ has geometric distribution with parameter $p$. Then*

*(i)* $E(X) = \dfrac{q}{p}$

*(ii)* $V(X) = \dfrac{q}{p^2}$

**Proof of (i)** (You are not responsible for this).

$$\begin{aligned} E(X) &= (0)(p) + (1)(qp) + (2)(q^2 p) \\ &\quad + \cdots + (k)(q^k p) + \cdots \\ &= p(q + 2q^2) + \cdots + kq^k + \cdots \end{aligned}$$

Now

$$\frac{x}{(1-x)^2} = x + \underset{\substack{\uparrow \\ \text{why?}}}{2x^2} + 3x^3 + \cdots + kx^k + \cdots$$

So

$$E(X) = p\left(\frac{q}{(1-q)^2}\right) = p\left(\frac{q}{p^2}\right) = \frac{q}{p}.$$

□

## 2 The Negative Binomial Distribution

Now suppose the couple decides they want more girls – say $r$ girls, so they keep having children until the $r$-th girl appears. Let

$$X = \text{the number of boys that precede the } r\text{-th girl.}$$

Find the probability distribution of $X$.

**Remark .** *Sometimes (e.g., pgs 13-14) it is better to write $X_r$ instead of $X$. Let's compute $P(X = k)$*

$$\underbrace{\_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_}_{k+r-1 \ \textit{people}} \ \underset{\substack{\uparrow \\ r\textit{-th} \\ \textit{girl}}}{\underline{G}}$$

*What do we have preceding the $r$-th girl. Of course we must have $r-1$ girls and since we are assuming $X = k$ we have $k$ boys so $k + r - 1$ children.*

All orderings of boys and girls have the same probability, so

$$P(X - k) = (?)P(\underbrace{B \cdots B}_{k-1} \ \underbrace{G \cdots G}_{r-1} \ G)$$

or

$$P(X = k) = (?)q^k p^{r-1} \cdot p = (?)q^k p^r$$

(?) is the number of words of length $k + r - 1$ in $B$ and $G$ using $k$ $B$'s (whence $r - 1$ $G$'s). Such a word is determined by choosing the $k$ slots occupied by the $B$'s. Hence there are $\binom{k+r-1}{k}$ such words so

$$(?) = \binom{k+r-1}{k}$$

$$\underbrace{\_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_}_{k+r-1 \ \text{slots}}$$

Choose $k$ slots and put in the $B$'s

$$\underline{B}\,\_\,\underline{B}\,\_\,\underline{B}\,\_\,\underline{B}.$$

Fill in $G$'s in the rest of the slots. So

$$P(X = k) = \binom{k+r-1}{k} p^r q^k.$$

So we have motivated the following:

**Definition.** *A discrete random variable $X$ is said to have* *<u>negative binomial distribution with parameters $r$ and $p$</u>* *if*

$$P(X = k) = \binom{k+r-1}{k} p^r q^k, \quad 0 \leq k < \infty.$$

*The text denotes this pmf by $nb(x; r, p)$, so*

$$nb(x; r, p) = \binom{k+r-1}{k} p^r q^k, \quad 0 \leq x < \infty.$$

**Proposition.** *Suppose $X$ has negative binomial distribution with parameters $r$ and $p$. Then*

*(i)* $E(X) = r\frac{q}{p}$

*(ii)* $V(X) = r\frac{rq}{p^2}$

### Waiting Time

The binomial, geometric and negative binomial distributions are all tied to repeating a given Bernoulli experiment (flipping a coin, having a child) infinitely many times. think of discrete time 0,1,2,3,... and we repeat the experiment at each of these discrete times – e.g., flip a coin every minute. Now you can do the following things:

1. Fix a time, say $n$, and let $X = \#$ of successes in that time period. Then $X \sim \text{Bin}(n, p)$. We should write $X_n$ and think of the <u>family</u> of random variables parameterized by the discrete time $n$ as the "binomial process"(see pg. 18 – the Poison process).

2. ((discrete) waiting time for the first success). Let $Y$ be the amount of time up to the time the first success occurs.

This is the geometric random variable. Why? Suppose we have in our boy/girl example

$$\underbrace{\frac{B}{0}\quad\frac{B}{1}\quad\frac{B}{2}\quad\frac{\phantom{B}}{3}\quad\frac{B}{k-1}}_{k\ \text{boys}}\quad\frac{G}{k}\ \text{minutes}.$$

So in this case,

$$X = \#\text{ of boys} = k.$$

but notice the girl arrived at the $k$-th minute so in the above

$$Y = K$$

so

$$Y = X.$$

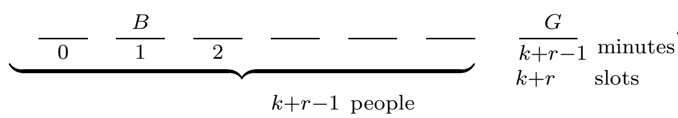
## 3 Waiting Time for $r$-th Success

Now let $Y_r =$ the waiting time up to the $r$-th success then there is a difference between $X_r$ and $Y_r$.

Suppose

$$X_r = k$$

so there are $k$ boys before the $r$-th girl arrives.




There are $k$ boys and $r$ girls (counting the last girl). The last girls arrives at time $k + r - 1$ so if $X_r = k$ then $Y_r = K = r - 1$ so

$$Y_r = X_r + r - 1.$$

## 4 The Poisson Distribution

For a change we won't start with a motivating example but will start with a definition.

**Definition.** *A discrete random variable $X$ is said to have Poisson distribution with parameter $\lambda$ if*

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad 0 \leq k < \infty.$$

*We will abbreviate this to*

$$X \sim P(\lambda).$$

*I will now try to motivate the formula which looks complicated.*

*Why is the fact of $e^{-\lambda}$ there? It is there to make the total probability equal to 1.*

$$\begin{aligned}\textit{Total Probability} &= \sum_{k=0}^{\infty} P(X = k) \\ &= \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}.\end{aligned}$$

*But from calculus*

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

*Total Probability* $= e^{-\lambda} \cdot e^{\lambda} = 1$ *as it has to be.*

**Proposition.** *Suppose*

*(i)* $E(X) = \lambda$

*(ii)* $V(X) = \lambda$

**Remark .** *It is remarkable that* $E(X) = V(X)$*.*

**Example 3.39**
Let $X$ denote the number of creatures of a particular type captured during a given time period. Suppose $X \sim P(4.5)$. Find $P(X = 5)$ and $P(X \leq 5)$.

**Solution**

$$P(X = 5) = e^{-4.5} \frac{(4.5)^5}{5!}$$

(just plug into the formula using $\lambda = 4.5$).

$$\begin{aligned} P(X \leq 5) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &\quad + P(X = 3) + P(X = 4) + P(X = 5) \\ &= e^{-\lambda} + e^{-\lambda}\lambda + e^{-\lambda}\frac{\lambda^2}{2} \\ &\quad + \underbrace{+e^{-\lambda}\frac{\lambda^3}{3!} + e^{-\lambda}\frac{\lambda^4}{4!} e^{-\lambda}\frac{\lambda^5}{5!}}_{\text{don't try to evaluate this}} \end{aligned}$$

# 5 The Poisson Process

A very important application of the Poisson distribution arises in counting the number of occurrences of a certain event in time $t$.

1. Animals in a trap.
2. Calls coming into a telephone switchboard.

Now we could let $t$ vary so we get a one-parameter family of Poisson random variable

$$X_t, \quad 0 \leq t < \infty.$$

Now a Poisson process is completely determined once we know its mean $\lambda$. So for each $t, X_t$ is a Poisson random variable. so

$$X_t \sim P(\lambda(t)).$$

So the Poisson parameter $\lambda$ is a function of $t$.

In the Poisson process one assume that $\lambda(t)$ is the simplest possible function of $t$ (aside from a constant function) namely the linear function

$$\lambda(t) = \alpha t.$$

Necessarily,

$$\alpha = \lambda(1) = \text{the average number of animals captured (or calls) in unit time.}$$

**Remark .** *In the test, page 124, the author proposes 3 axioms on a one parameter family of random variables $X_t$ so that $X_t$ is a Poisson process, i.e.,*

$$X_t \sim P(\alpha t).$$

**Example .** *(from an earlier version of the text). The number of tickets issued by a meter reader cam be modeled by a Poisson process with a rate of 10 tickets every two hours.*

*(a) What is the probability that exactly 10 tickets are given out during a particular 12 hour period.*

**Solution**

*We want $P(X_{12} = 10)$. First find*

$$\alpha = \textit{average number of tickets per unit time.}$$

*so*

$$\alpha = \frac{10}{2} = 5.$$

*So*

$$X_t \sim P(5t)$$

*so*

$$X_{12} \sim P((15)(12)) = P(60)$$

$$P(X_{12} = 10) = e^{-\lambda} \frac{\lambda^{10}}{(10)!} = e^{-60} \frac{(60)^{10}}{(10)!}.$$

*(b) What is the probability that at least 10 tickets are given out during a 12 hour time period.*

We want

$$\begin{aligned}
P(X_{12} \geq 10) &= 1 - P(X \leq 9) \\
&= 1 - \sum_{k=0}^{9} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= 1 - \underbrace{\sum_{k=0}^{9} e^{-60} \frac{(60)^k}{k!}}_{\substack{\text{not something you want to} \\ \text{try to evaluate by hand}}}
\end{aligned}$$

## 6 Waiting Time

Again there are waiting time random variables associated to the Poisson process.

Let

$$Y = \text{waiting time until the first animal is caught in the trap}$$

and

$$Y_r = \text{waiting time until the } r\text{-th animal is caught in the trap}$$

Now $Y$ and $Y_r$ are continuous random variables which we are about to student. $Y$ is exponential and $Y_r$ has a special kind gamma distribution.