Lecture 3

Probability Computations

Pg 67, #42 is one of the hardest problems in the course. The answer is a simple fraction so there should be a simple way to do it. I don't know a simple way - I used the formula for $P(A \cup B \cup C)$, text pg 56 or Lecture 1, pg 30.

If you find a simple way show it to me or your TA and you will get a "gold star". We will remember it when it comes to final grade time.

The rest of this lecture will be about
Bridge Hands and Poker Hands

Bridge Hands

If you play bridge you get dealt a hand of 13 cards

Let S be the set of all bridge hands (so our "experiment" is dealing 13 cards).

Since the order in which you receive the cards doesn't count (it never does in card games)

$$\#(S) = \binom{52}{13}$$

= the number of 13 element subsets of a 52 element set.

We will now compute the probability of certain bridge hands.

1. Let A = the hand is all hearts

What is $\#(A)$? We use the "principle of restricted choice". Our choice is restricted to the subset of hearts – we have to choose 13. There are 13 hearts so we have $\binom{13}{13} = 1$ hands.

So $$P(A) = \frac{\#(A)}{\#(S)} = \frac{1}{\binom{52}{13}}$$

(A is very unlikely)

2. Let B = there are no hearts

warning = dangerous, turn

$$B \neq A'$$

A' = at least one nonheart.

What is $\#(B)$?

Once again we use the "principle of restricted choice" (I made up this name, it isn't in common usage). We have to choose 13 nonhearts. There are $52-13 = 39$ nonhearts so we have to choose 13 things from 39 things so

$$\#(B) = \binom{39}{13}$$

So

$$P(B) = \frac{\binom{39}{13}}{\binom{52}{13}}$$

Poker Hands

If you ploy poker you get dealt a hond of 5 cards (or sometimes 7 cards). For the next few examples we will assume we are playing "5-cord poker". There is also the role of aces. In the text pg 67, #43 aces can be "either high or low". This means ~~that~~ you can count ~~them~~ as 1's (ie lower than anything else) or higher than anything else so we have in order

A 2 3 4 5 6 7 8 9 10 J Q K A

ace — jack — queen — king — ace

There are $\binom{52}{5}$ (five-card) poker hands.

A "Straight"

1. A = all five cards are consecutive = a "straight"

e.g. 56789

or A2345 aces are low

or 10JQKA aces are high

Find $P(A)$, this is Problem 43 from the text.

There is one observation you need —

a straight is almost determined by its bottom (ie lowest) card.

You need one more subobservation: ~~the cards~~ cards, J, Q, K cannot be bottom cards for a straight (because there aren't enough cards above them).

The last straight is

10 J K Q A ← aces are high

Also A can be a bottom card because aces are low

A 2 3 4 5

So there are 13 − 3 = 10 bottom cards so there are 10 different kinds of straights — so at first glance one might think

there are 10 straights.
BUT, let's consider
A2 3 4 5. We haven't taken account of the SUITS of the cards.
The A could be any one of four suits, for each of these the 2 could be any one of four suits so there are 4^5 straights of the form A2345. Hence

$$\#(A) = (\underset{\substack{\uparrow \\ \text{lowest card}}}{\underline{10}})(\underset{\substack{\uparrow \\ \text{suit of each card}}}{\underline{4^5}})$$

so

$$P(A) = \frac{(10)(4^5)}{\binom{52}{5}}$$

2 A "Flush"

A "flush" is a poker hand in which all cards are of the same suit (spade, heart, diamond or club).

Let B = set of all flushes.

First – how many hands are there that are all hearts?

We use the principle of restricted choice. There are 13 hearts, we have to choose 5. So there are $\binom{13}{5}$ such hands so we have

$$\#(B) = \underbrace{4}_{\text{choose a suit}} \underbrace{\binom{13}{5}}_{\text{pick 5 cards from it}} = (4)\binom{13}{5}$$

so $P(B) = \dfrac{(4)\binom{15}{5}}{\binom{52}{5}}$

3. A "Straight Flush"

Lets combine 2. and 3. so

C = set of all straights so that all the cards have the same suit

To compute $\#(C)$ first

pick the lowest card 10 ways

then
pick the suit that all five cards have 4 ways

So $\#(C) = (10)(4) = 40$

and $P(C) = \dfrac{40}{\binom{52}{5}}$ = Very a small number

4. A Full House

A full house is a poker hand which consists of 3 of one kind and 2 of a different kind e.g JJJ KK

Let D = set of full houses.

Here is how we compute $\#(D)$

1. Pick an <u>ordered</u> pair of kinds e.g. J or K.

2. Pick 3 of the first and 2 of the second

so in the above case
we get JJJ KK.

Key test to perform

Were we right when we
said ORDERED pair above?
So let's test, reverse the
order K, J and check
if we get a different hand.
If we do we were right
to say ORDERED pairs.
If we don't we were wrong
and we have to replace ORDERED
pairs in 1. by UNORDERED
pairs.

Okay if we take the pair K, J and do 2. we get the hand KKKJJ

Are KKKJJ and JJJKK different? <u>yes</u>, the first beats the second so we were right to pick ordered pairs.

Now lets finish the job

1. There are 13 kinds, an ordered pair of kinds is a 2 permutation of the 13 element set of kinds

so $\#(1.) = P_{2,13} = \dfrac{(13)!}{(11)!}$

$= (13)(12)$

2. Now there are $\binom{4}{3}$ to pick the first kind (say the first kind is a jack so we have to pick 3 of the 4 jacks) and $\binom{4}{2}$ ways to pick the second kind so

$$\#(D) = \underbrace{(13)(12)}_{2\text{ kinds}} \binom{4}{3}\binom{4}{2}$$

so

$$P(D) = \frac{(13)(12)\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

4. Two Pair

In poker "two pair" is a hand which consists of 2 of one kind, 2 of a second (<u>different</u>) kind and one more of a third kind (<u>different</u> from the other two)

e.g. J J K K 10

Let E be the set of two pair hands. Here is how we compute #(E)

1. Pick an $\left\{ \begin{array}{l} \text{~~ordered~~} \\ \text{unordered} \end{array} \right.$ pair of kinds,

   e.g. J, K.

2. Pick a third kind.

3. Pick two each of the first two kinds and one of the

third kind.

$64,000 Question

In 1. do we pick an ordered pair or an unordered pair?

Answer Do the test?

The pair J, K ⟶ JJKK

The pair K, J ⟶ KKJJ

In poker JJKK and KKJJ are the SAME so J, K and K, J give the SAME hand

so order does not matter

so we pick an unordered pair

Now we compute #(E)

1. There are 13 kinds
We want the number
of <u>unordered</u> pairs. It
is $\binom{13}{2} = \frac{(13)(12)}{2}$

$= \frac{1}{2}$ the number we got
for 1. in the full house case

2. We have chosen two
kinds. There are $13 - 2 = 11$
left. We have to pick one
so $\binom{11}{1} = 11$.

3. $\binom{4}{2} = 6$ for the first kind.

$\binom{4}{2} = 6$ for the second kind

(there is no first or second but it does matter the numbers are the same)

$\binom{4}{1} = 4$ for the third kind.

$$\#(E) = \binom{\frac{(13)(12)}{2}}{}\cdot\binom{11}{}\binom{4}{2}\binom{4}{2}\binom{4}{}$$

so

$$P(E) = \frac{\left(\frac{(13)(12)}{2}\right)(11)(6)(6)(4)}{\binom{52}{5}}$$