

Lecture 8

The Geometric Distribution

The geometric distribution is a special case of negative binomial, it is the case $r=1$. It is so important we give it special treatment.

Motivating example

Suppose a couple decides to have children until they have a girl. Suppose the probability of having a girl is p . Let

$X =$ the number of boys that precede the first girl

Find the probability distribution of X . First X could have any possible whole number value (although $X=1,000,000$ is very unlikely).

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$$P(X=k) = P(\underbrace{B \ B \ B \ \dots \ B}_k \ G)$$

\uparrow
 p

$$= q^k p \quad (\text{where } q = 1-p)$$

We have suppose births are independent.

We have motivated

Definition

Suppose a discrete random variable X has the following pmf

$$P(X=k) = q^k p, \quad 0 \leq k < \infty$$

The X is said to have geometric distribution with parameter p .

Remark

Usually this is developed by replacing "having a child" by a Bernoulli experiment and having a girl by a "success" (PC). I could have used coin flips.

The Negative Binomial

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Distribution

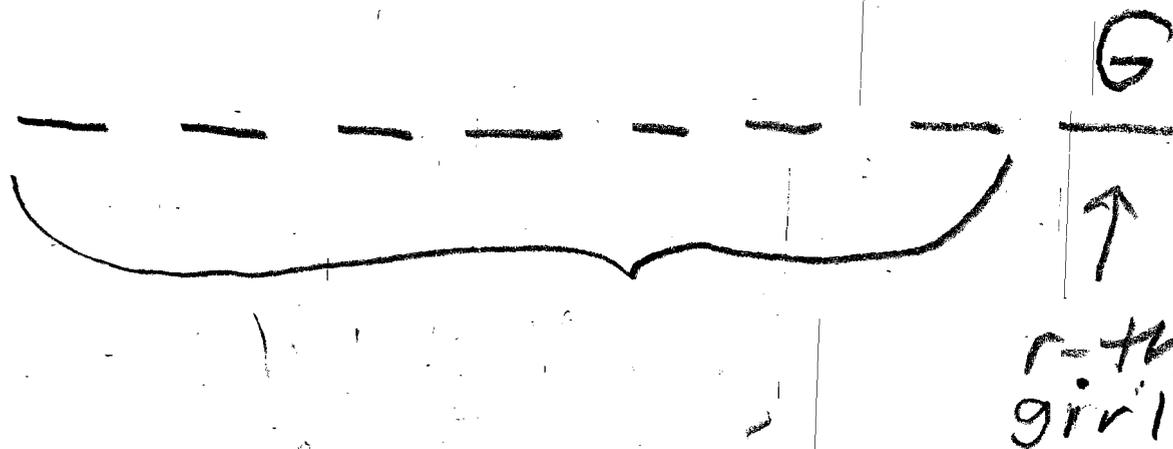
Now suppose the couple decides they want more girls - say r girls; so they keep having children until the r -th girl appears. Let X = the number of boys that precede the r -th girl.

Find the probability distribution of X .

Remark Sometimes (eg pg 13-14) it is better to write X_r instead of X .

Let's compute $P(X=k)$

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What do we have preceding the r -th girl. Of course we must have $r-1$ girls and since we are assuming $X=k$ we have k boys so $k+r-1$ children.

All orderings of boys and girls have the same probability so

$$P(X=k) = (?) P(\underbrace{B \dots B}_{k-1} \underbrace{G \dots G}_{r-1} G)$$

or

$$P(X=k) = \binom{?}{k} q^k \cdot p^{r-1} \cdot q = \binom{?}{k} q^k p^r$$

$\binom{?}{k}$ is the number of

words of length $k+r-1$

in Board G using

k B's (whence $r-1$ G's).

Such a word is determined

by choosing the slots occupied

by the B's so there are

$\binom{k+r-1}{k}$ words so

$$P(X=k) = \binom{k+r-1}{k} p^r q^k$$

So we have motivated
the following

Definition

A discrete random variable
 X is said to have negative
binomial distribution with
parameters r and p if

$$P(X=k) = \binom{k+r-1}{k} p^r q^k, \quad 0 \leq k < \infty$$

The text denotes this pmf
by $nb(x; r, p)$. so

$$nb(x; r, p) = \binom{x+r-1}{k} p^r q^x, \quad 0 \leq x < \infty.$$

Proposition

Suppose X has negative binomial distribution with parameters r and p . Then

$$(i) E(X) = r \frac{q}{p}$$

$$(ii) V(X) = \frac{r q}{p^2}$$

Waiting Times

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The binomial, geometric and negative binomial distributions are all tied to repeating a given

Bernoulli experiment

(flipping a coin, having a child) infinitely many times.

Think of discrete time

0, 1, 2, 3, ...

and we repeat the experiment at each of these discrete times.

— Eg. flip a coin every minute

Now you can do the following 11.
things

1. Fix a time say n and
let $X = \#$ of successes
in that time period. Then
 $X \sim \text{Bin}(n, p)$. We should
write X_n and think of
the family of random variables
parametrized by the discrete time n
as the "binomial process"
(see pag 18 — the Poisson process).

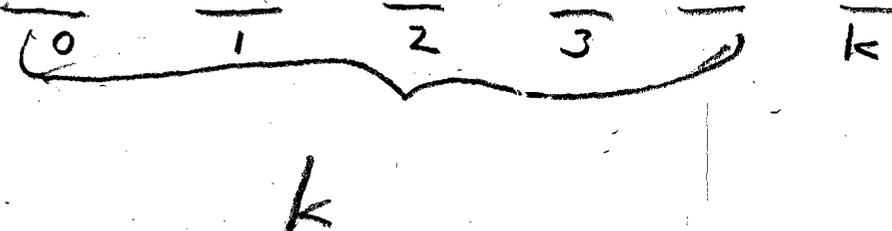
2. ((discrete) waiting time for
the first success)

Let Y be the amount of
time up to the time the
first success occurs.

This is the geometric random variable. Why?

Suppose we have in our boy/girl example

B	B	B	B	B	G
$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	\overline{k}	\overline{k}



So in this case

$X = \# \text{ of boys} = k$

$Y = \text{waiting time} = k$

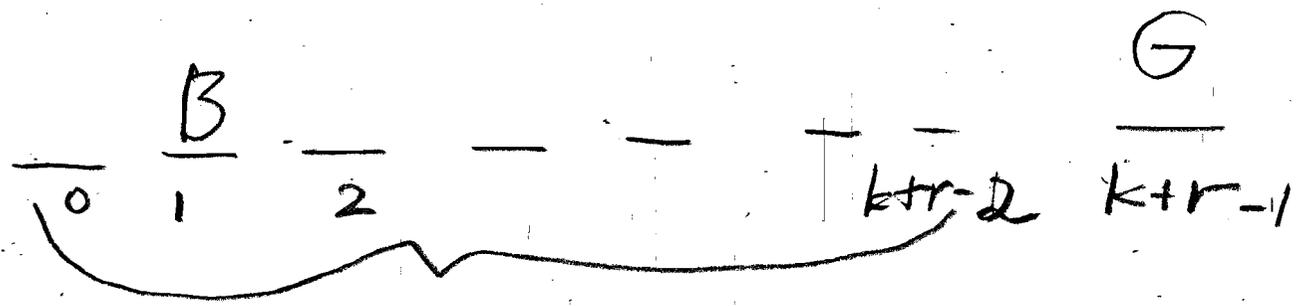
so $Y = X$

3 - Waiting time for r-th success

Now let Y_r = the waiting time up to the r-th success then there is a difference between X_r and Y_r

Suppose $X_r = k$

so there are k boys before the r-th girl arrives



k B's r-1 G's so k+r-1 slots

But start at 0 so the last slot is $k+r-2$ so

$$Y_r = X_r + r - 1$$

The Poisson Distribution

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For a change we won't start with a motivating example but will start with the definition.

Definition

A discrete random variable X is said to have Poisson distribution with parameter λ

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad 0 \leq k < \infty$$

We will abbreviate this to

$$X \sim P(\lambda).$$

I will now try to motivate the formula which looks complicated.

Why is the factor of $e^{-\lambda}$ here? It is there to make total probability equal to 1.

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$$\begin{aligned} \text{Total Probability} &= \sum_{k=0}^{\infty} P(X=k) \\ &= \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \end{aligned}$$

But from calculus

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Total Probability = $e^{-\lambda} \cdot e^{\lambda} = 1$
as it has to be.

Proposition

Suppose $X \sim \text{PC}(\lambda)$. Then

(i) $E(X) = \lambda$

(ii) $V(X) = \lambda$

Remark It is remarkable that $E(X) = V(X)$.

Example 3.39

Let X denote the number of creatures of a particular type captured during a given time period. Suppose

$X \sim P(4.5)$. Find $P(X=5)$

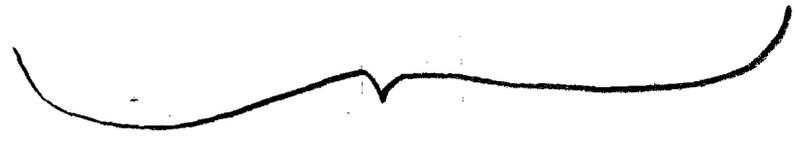
and $P(X \leq 5)$

Solution

$$P(X=5) = e^{-4.5} \frac{(4.5)^5}{5!}$$

(just plug into the formula using $\lambda = 4.5$)

$$\begin{aligned}
 P(X \leq 5) &= P(X=0) + P(X=1) + P(X=2) \\
 &\quad + P(X=3) + P(X=4) + P(X=5) \\
 &= e^{-\lambda} + e^{-\lambda} \lambda + e^{-\lambda} \frac{\lambda^2}{2} \\
 &\quad + e^{-\lambda} \frac{\lambda^3}{3!} + e^{-\lambda} \frac{\lambda^4}{4!} + e^{-\lambda} \frac{\lambda^5}{5!}
 \end{aligned}$$



don't try to evaluate this.

The Poisson Process

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A very important application of the Poisson distribution arises in counting the number of occurrences of a certain event in time t

1. Animals in a trap
2. Calls coming into a telephone switchboard

Now we could let t vary so we get a one-parameter

family of Poisson random variables X_t , $0 \leq t < \infty$.

Now a Poisson process is completely determined once we know its mean λ .

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So for each t , X_t is
a Poisson random variable. so
 $X_t \sim P(\lambda(t))$

So the Poisson parameter λ
is a function of t .

In the Poisson process one
assume that $\lambda(t)$ is the
simplest possible function of
 t (aside from a constant function)
namely a linear function

$$\lambda(t) = \alpha t.$$

Necessarily
 $\alpha = \lambda(1) =$ the average number
of observations in
unit time.

Remark

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In the text, page 124, the author proposes 3 axioms on a one parameter family of random variables X_t so that X_t is a Poisson process i.e

$$X_t \sim P(\lambda t)$$

Example (from an earlier version of the text)

The number of tickets issued by a meter reader can be modelled by a Poisson process with a rate of 10 tickets every two hours.

(a) What is the probability that exactly 10 tickets are given out during a particular 12 hour period.

Solution

We want $P(X_{12} = 10)$.

First find $\alpha =$ average # of tickets by unit time

$$\text{so } \alpha = \frac{10}{2} = 5$$

$$\text{so } X_t \sim P(5t)$$

so $X_{12} \sim P((5)(12)) = P(60)$

$$P(X_{12} = 10) = \frac{e^{-60} 60^{10}}{(10)!}$$

$$= e^{-60} \frac{(60)^{10}}{(10)!}$$

(b) What is the probability that at least 10 tickets are given out during a 12 hour time period.

We want

$$\begin{aligned} P(\tilde{X}_{12} \geq 10) &= 1 - P(X \leq 9) \\ &= 1 - \sum_{k=0}^9 e^{-\lambda} \frac{\lambda^k}{k!} \\ &= 1 - \sum_{k=0}^9 e^{-60} \frac{(60)^k}{k!} \end{aligned}$$

not something you
want to try to
evaluate by hand.

Waiting Times

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Again there are waiting time random variables associated to the Poisson process.

Let Y = waiting time until the first animal is caught in the trap

and Y_r = waiting time until the r -th animal is caught in the trap

Now Y and Y_r are continuous random variables which we are about to study. Y is exponential and Y_r has a special kind gamma distribution.