Lecture 14

The Gamma Distribution and its Relatives

The gamma distribution is a continuous distribution depending on two parameters, α and β. It gives rise to three special cases

1. The exponential distribution ($\alpha = 1, \beta = \frac{1}{\lambda}$)
2. The r-Erlang distribution ($\alpha = r, \beta = \frac{1}{\lambda}$)
3. The chi-squared distribution ($\alpha = \nu/2, \beta = 2$)

The Gamma Distribution

Definition

A continuous random variable X is said to have gamma distribution with parameters α and β, both positive, if

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

What is $\Gamma(\alpha)$?

$\Gamma(\alpha)$ is the <u>gamma function</u>, one of the most important and common functions in advanced mathematics.

If α is a positive integer n then

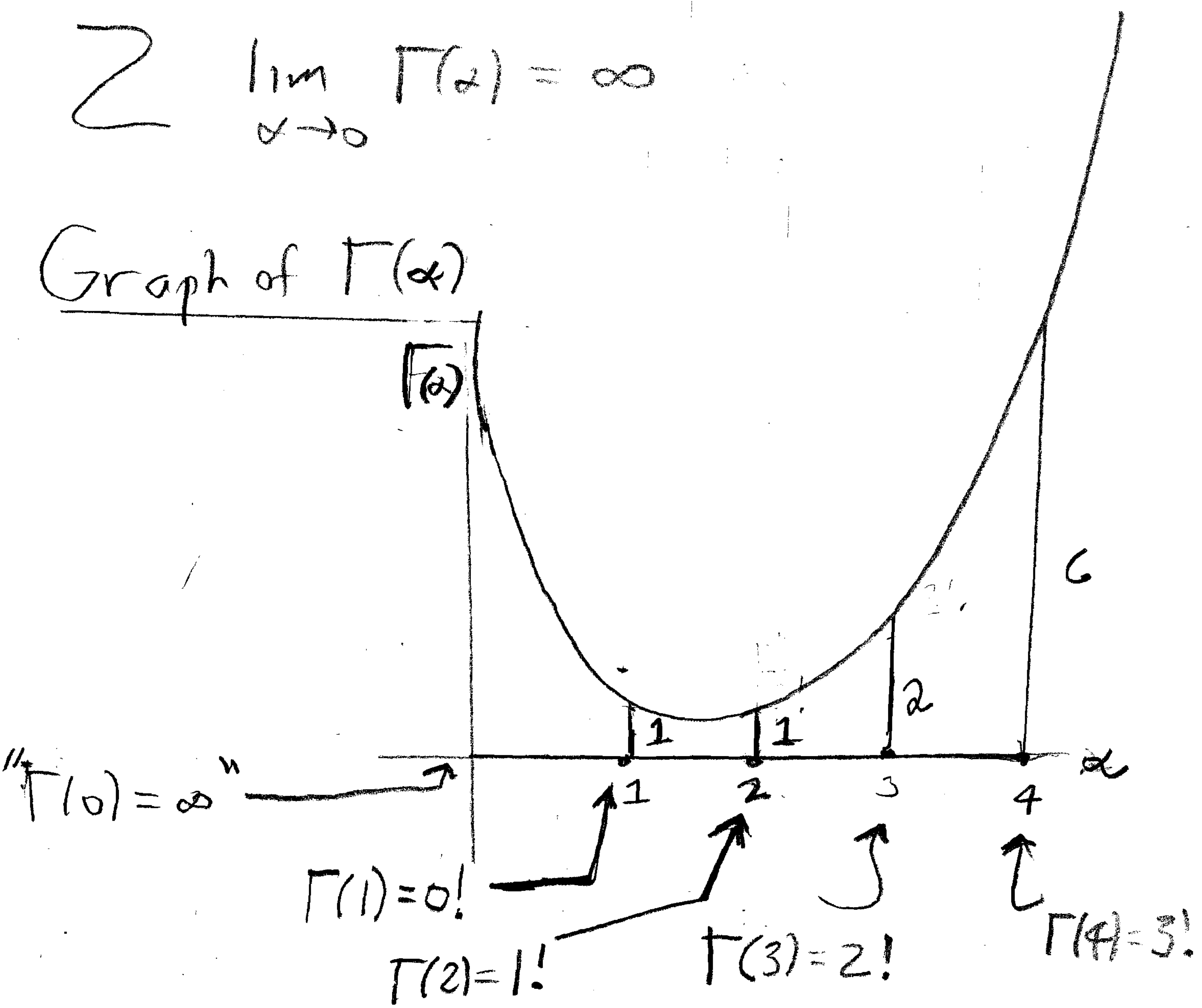
$$\Gamma(n) = (n-1)!$$

(see page 17)

So $\Gamma(\alpha)$ is an interpolateon of the factorial functeon to all real numbers.

2 $\lim_{\alpha \to 0} \Gamma(\alpha) = \infty$

Graph of $\Gamma(\alpha)$



I will say more about the gamma function later. It isn't that important for Stat 400, here it is just a constant chosen so that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

The key point of the gamma distribution is that it is of the form

$$(\text{constant})(\text{power of } x)\, e^{-cx}, \quad c>0.$$

The r-Erlang distribution from Lecture 13 is almost the most general gamma distribution.

The only special feature here is that α is a whole number r.

Also $\beta = \frac{1}{\lambda}$ where λ is the Poisson constant.

Comparison

Gamma distribution

$$\left(\frac{1}{\beta}\right)^{\alpha} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

r-Erlang distribution

$\alpha = r, \ \beta = \frac{1}{\lambda}$

$$\lambda^{r} \frac{1}{(r-1)!} x^{r-1} e^{-\lambda x}$$

Proposition

Suppose X has gamma distribution with parameters α and β then

(i) $E(X) = \alpha\beta$

(ii) $V(X) = \alpha\beta^2$

so for the r-Erlang distribution

(i) $E(X) = \dfrac{r}{\lambda}$

(ii) $V(X) = \dfrac{r}{\lambda^2}$

As in the case of the normal distribution we can compute general gamma probabilities by <u>standardizing</u>.

Definition

A gamma distribution is said to be standard if $\beta = 1$. Hence the pdf of the standard gamma distribution is

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The cdf of the standard

gamma function is called the <u>Incomplete gamma function</u> (divided by $\Gamma(\alpha)$)

$$F(x) = \frac{1}{\Gamma(\alpha)} \int_0^x x^{\alpha-1} e^{-x} \, dx$$

(see page 13 for the actual gamma function)

It is tabulated in the text Table A.4 for some (integral values of α)

<u>Proposition</u>

Suppose X has gamma distribution with parameters α and β. Then $Y = \frac{X}{\beta}$ has standard gamma distribution.

Proof

We can prove this, $y = \frac{x}{\beta}$ so $x = \beta y$.

Now $f_X(x)\,dx = \frac{1}{\beta^\alpha}\frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\,dx$

Now substitute $x = \beta y$ to get

$$f_Y(y)\,dy = \frac{1}{\beta^\alpha}\frac{1}{\Gamma(\alpha)}(\beta y)^{\alpha-1} e^{-\frac{\beta y}{\beta}}\,d(\beta y)$$

$$= \frac{1}{\cancel{\beta^\alpha}}\frac{1}{\Gamma(\alpha)} \cancel{\beta^{\alpha-1}} y^{\alpha-1} e^{-y} \cancel{\beta}\,dy$$

$$= \underbrace{\frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}\,dy}_{\text{standard gamma}}$$

$\square$

Example 4-24 (cut down)

Suppose X has gamma distribution with parameters $\alpha = 8$ and $\beta = 15$. Compute

$$P(60 \le X \le 120)$$

Solution

standardize, divide EVERYTHING by $\beta = 15$

$$P(60 \le X \le 120) = P\left(\frac{60}{15} \le \frac{X}{15} \le \frac{120}{15}\right)$$

$$= P(4 \le Y \le 8) = F(8) - F(4)$$

from table A-4

$$= .547 - .051 = .496$$

The Chi-Squared Distribution

Definition

Let ν (Greek letter nu) be a positive real number. A continuous random variable X is said to have chi-squared distribution with ν degrees of freedom if X has gamma distribution with $\alpha = \nu/2$ and $\beta = 2$.

Hence

$$f(x) = \begin{cases} \dfrac{1}{2^{\nu/2}\,\Gamma(\nu/2)}\, x^{\nu/2 - 1} e^{-\frac{x}{2}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

We will write $X \sim \chi^2(\nu)$.

capital chi

The reason the chi-squared distribution is that if

$$Z \sim N(0,1) \text{ then } X = Z^2 \sim \chi^2(1)$$

and if $Z_1, Z_2, \ldots, Z_m$ are independent random variables the

$$Z_1^2 + Z_2^2 + \cdots Z_m^2 \sim \chi^2(m)$$

(later).

<u>Proposition</u> (<u>Special case of pg 6</u>)

If $X \sim \chi^2(\nu)$ then

(i) $E(X) = \nu$

(ii) $V(X) = 2\nu$

Appendix

The Gamma Functeon

Definiteon

For $\alpha > 0$, the gamma functeon $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha - 1} e^{-x} dx$$

Remark 1. It is more natural to write

this is the variable

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha} e^{-x} \frac{dx}{x}$$

but I won't explain why unless you ask.

<u>Remark 2</u>. In the <u>complete gamma function</u> we integrate from 0 to <u>infinity</u> whereas for the <u>incomplete gamma function</u> we integrate from 0 to <u>x</u>.

$$F(x;\alpha) \int_0^x y^{\alpha-1} e^{-y} dx .$$

Thus

$$\lim_{x\to\infty} F(x;\alpha) = \Gamma(\alpha) .$$

Remark 3. Many of the "special functions" of advanced mathematics and physics e.g. Bessel functions, hypergeometric functions ... arise by taking an elementary function of x depending on a parameter (or parameters) and integrating with respect to x leaving a function of ~~the~~ parameter

Here the elementary function is $x^{\alpha-1}e^{-x}$. We "integrate out the x" leaving a function of α.

Lemma

$$\Gamma(1) = 1$$

Proof

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left(-e^{-x}\right)\Big|_0^\infty = 1$$

$\square$

The Functional Equation for the Gamma Function

Theorem

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha) \qquad , \alpha > 0$$

Proof Integrate by parts

$$\Gamma(\alpha+1) = \int_0^\infty \overset{u}{x^{\alpha}}\ \overset{dv}{e^{-x}\,dx}$$

$$= \left(-\underset{u}{x^{\alpha}}\,\underset{v}{e^{-x}}\right)\Big|_0^\infty - \int_0^\infty \overset{du}{\alpha x^{\alpha-1}}\ \overset{v}{\left(-e^{-x}\right)}\,dx$$

$$= \alpha \int_0^\infty x^{\alpha-1} e^{-x}\,dx$$

$\square$

<u>Corollary</u> If n is a whole number

$$\Gamma(n) = (n-1)!$$

<u>Proof</u> I will show you $\Gamma(4) = 3!$

$$\Gamma(4) = \Gamma(3+1) = 3\,\Gamma(3)$$

$$= 3\,\Gamma(2+1) = (3)(2)\,\Gamma(2)$$

$$= (3)(2)\,\Gamma(1+1) = (3)(2)(1)\,\Gamma(1)$$

$$= (3)(2)(1)$$

In general you use induction. $\square$

We will need Γ(half integers)

eg $\Gamma\left(\frac{5}{2}\right)$.

<u>Theorem</u>

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

I won't prove this. Try it.

$$\Gamma(3/2) = \Gamma(\tfrac{1}{2}+1) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(5/2) = \Gamma(3/2+1) = \frac{3}{2}\Gamma(\tfrac{3}{2}) = (\tfrac{3}{2})(\tfrac{1}{2})\sqrt{\pi}$$

In general

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(1)(3)(5)\cdots(2n-1)}{2^n}\sqrt{\pi}$$

For statistics we will need only $\Gamma(\text{integer}) = (\text{integer}-1)!$

and $\Gamma\left(\frac{\text{odd integer}}{2}\right) = \text{above}$