

# Lecture 15

## Pairs of Discrete Random Variables

Today we start Chapter 5. The transition we are making is like going from one variable calculus to vector calculus. We should really think of vectors  $(X, Y)$  of random variables.

So suppose  $X$  and  $Y$  are discrete random variables defined on the same sample space  $S$ .

Definition The joint probability mass function, joint pmf,  $P_{X,Y}(x,y)$ , is defined by  
and  
$$P_{X,Y}(x,y) = P(X=x, Y=y).$$

## Example

A fair coin is tossed three times.

Let

$X = \#$  of head on first toss

$Y = \text{total } \# \text{ of heads}$

As usual

$$S = \left\{ \begin{array}{l} HHH, HHT, HTH, HTT \\ THH, THT, TTH, TTT \end{array} \right\}$$

We want to compute

$$P_{X,Y}(x,y) = P(X=x, Y=y)$$

$$= P((X=x) \cap (Y=y))$$

$$= P(X=x) P(Y=y | X=x)$$

We will record the results in a matrix which we will now compute

$X \backslash Y$	0	1	2	3
0		↑		
1				

$P(X=0, Y=1)$

First column ( $y=0$ )

Let's compute (upper left,  $x=0$ )

$$P_{X,Y}(0,0) = P((X=0) \cap (Y=0))$$

$$= P(Y=0) = P(TTT) = \frac{1}{8}$$

↑ (because  $Y=0 \Rightarrow X=0$ )

Now lower left ( $x=1$ )

$$P_{X,Y}(1,0) = P(X=1, Y=0) = 0$$

Move to the 2<sup>nd</sup> column ( $y=1$ )

$$P_{X,Y}(0,1) = P(X=0, Y=1) \quad (\text{top entry } X=0)$$

This is harder

4

$$P(X=0, Y=1) = P(\text{T on first and exactly 1 head total}) \\ = P(\text{THT}) + P(\text{TTH}) = \frac{2}{8}$$

The bottom entry of the second column is

$$P(X=1, Y=1) = P(\text{HTT}) = \frac{1}{8}$$

Third column (y=2)

$$P(X=0, Y=2) = P(\text{THH}) = \frac{1}{8}$$

$$P(X=1, Y=2) = P(\text{HTH}) + P(\text{HHT}) \\ = \frac{2}{8}$$

Fourth column (y=3)

$$P(X=0, Y=3) = 0$$

$$P(X=1, Y=3) = P(\text{HHH}) = \frac{1}{8}$$

The table for the joint pmf

$$P_{X,Y}(x,y)$$

$X \backslash Y$	0	1	2	3
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

(\*)

Check that the total probability is 1.

The joint pmf has a huge amount of information in it. In particular it contains the pmf  $P_X(x)$  of  $X$  and  $P_Y(y)$  of  $Y$ .

So how do we recover  $P(Y=1)$  from the table above. The event  $(Y=1)$  is the union of the two events  $(X=0, Y=1)$  and  $(X=1, Y=1)$ . These two are mutually exclusive

So

6

$$\begin{aligned} P(Y=1) &= P(X=0, Y=1) + P(X=1, Y=1) \\ &= \frac{2}{8} + \frac{1}{8} = \frac{3}{8} \end{aligned}$$

= the sum of the entries in the second column (ie. the column corresponding to  $y=1$ ).

How about  $P(X=1)$ ?

We have an equality of events

$$\begin{aligned} \{X=1\} &= (X=1, Y=0) \cup (X=1, Y=1) \\ &\quad \cup (X=1, Y=2) \cup (X=1, Y=3) \\ &= 0 + \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{1}{2} \end{aligned}$$

= the sum of the entries in the second row (corresponding to  $x=1$ )

So we see we recover

$P_Y(y)$  by taking column sums

and  $P_X(x)$  by taking row sums.

## Marginal Distributions

We can express the above nicely by expanding the table (\*) "adding margins".

Table (\*) with margins added

$x \backslash y$	0	1	2	3	
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	

(\*)

# The \$64,000 question

8

How do you fill in the margins?

There is only one reasonable way to do this — put the row sums in the right margin and the column sums in the bottom margin

Table (\*\*) with the margins filled in

$x \backslash y$	0	1	2	3	
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{1}{2}$
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	

(\*\*\*)

The right margin tells us the pmf of  $X$  and the bottom margin tells us the pmf of  $Y$ .

$x$	$\frac{1}{2}$
0	$\frac{1}{2}$
1	$\frac{1}{2}$

$y$	0	1	2	3
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⋮

$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
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So we have

$x$	0	1
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}$

and

$y$	0	1	2	3
$P(Y=y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$X \sim \text{Bin}(1, \frac{1}{2})$$

$$Y \sim \text{Bin}(3, \frac{1}{2})$$

For this reason, given the pair  $(X, Y)$  the

p.m.f's  $P_X(x)$  and  $P_Y(y)$

are called the marginal distributions.

To state all this correctly we have

### Proposition

$$(i) \quad P_X(x) = \sum_{\text{all } y} P_{X,Y}(x,y) \quad (\text{row sum})$$

$$(ii) \quad P_Y(y) = \sum_{\text{all } x} P_{X,Y}(x,y) \quad (\text{column sum})$$

So you "sum away" one variable leaving a function of the remaining variable.

# Combining Discrete Random Variables

11.

Suppose  $X$  and  $Y$  are discrete random variables defined on the same sample space. Let  $h(x, y)$  be a real-valued function of two variables. We want to define a new random variable

$$W = h(X, Y)$$

## Examples

We will start with the pair  $(X, Y)$  from our basic example.

The key point is that a function of a pair of random variables is again a random variable

We will need only the joint pmf 12

$x \setminus y$	0	1	2	3
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

(\*)

(i)  $h(x, y) = x + y$  so  $W = X + Y$

We see that the possible values of the sum are 0, 1, 2, 3, 4

(since they are the sums of the possible values of  $X$  and  $Y$ ).

We need to compute their probabilities. How do you compute

$$P(W=0) = P(X+Y=0)?$$

Answer Find all the pairs  $x$  and  $y$  that add up to zero, take the probability of each such pair and add the resulting probabilities.

But  $X+Y=0 \Leftrightarrow X=0$  and  $Y=0$  13  
so there is only one such pair  $(0,0)$   
and (from the joint pmf  $(*)$ )

$$P(X=0, Y=0) = \frac{1}{8}$$

Hence

$$P(W=0) = P(X=0, Y=0) = \frac{1}{8}$$

$$P(W=1) = P(X+Y=1)$$

$$= P(X=0, Y=1) + P(X=1, Y=0)$$

$$= \frac{2}{8} + 0 = \frac{2}{8}$$

$$P(W=2) = P(X=0, Y=2)$$

$$+ P(X=1, Y=1)$$

$$= \frac{2}{8}$$

Similarly

14

$$P(W=3) = \frac{2}{8} \quad \text{and} \quad P(W=4) = \frac{1}{8}$$

So we get for  $W = X+Y$

$w$	0	1	2	3	4
$P(W=w)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

(L)

(check that the total probability is 1)

Remark Technically the rule given in the "Answer" above is the definition of  $W = X+Y$  as a random variable but as usual the definition is forced on us.

$$(ii) \quad h(x,y) = xy \quad \text{so} \quad W = XY$$

The possible values of  $W$

(the products of values of  $X$  with those of  $Y$ ) are 0, 1, 2, 3.

We now compute their probabilities.

$$\underline{P(W=0)}$$

We can get 0 as a product  $xy$  if either  $x=0$  or  $y=0$  so we have

$$\begin{aligned} P(W=0) &= P(XY=0) \\ &= P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2) \\ &\quad + P(X=0, Y=3) + P(X=1, Y=0) \\ &= \frac{1}{8} + \frac{2}{8} + \frac{1}{8} + 0 + 0 = \frac{1}{2} \end{aligned}$$

$$\underline{P(W=1)}$$

$$P(W=1) = P(X=1, Y=1) = \frac{1}{8}$$

$$\underline{P(W=2)}$$

$$P(W=2) = P(X=1, Y=2) = \frac{2}{8}$$

$$\underline{P(W=3)}$$

$$P(W=3) = P(X=1, Y=3) = \frac{1}{8}$$

W	0	1	2	3
$P(W=w)$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

(check that the total probability is 1)

(iii)  $h(x, y) = \max(x, y) =$  the bigger of  $x$  and  $y$

$$\text{so } W = \max(X, Y)$$

Remark The max function doesn't turn up in vector calculus but it turns up a lot in statistics in advanced mathematics and real life.

The possible values of  $\max(x, y)$  are 0, 1, 2, 3.

$$\underline{P(W=0)}$$

$$\begin{aligned} P(W=0) &= P(\text{Max}(X, Y)=0) \\ &= P(X=0, Y=0) = \frac{1}{8} \end{aligned}$$

$$\underline{P(W=1)}$$

$$\begin{aligned} P(W=1) &= P(\text{Max}(X, Y)=1) \\ &= P(X=0, Y=1) + P(X=1, Y=0) \\ &\quad + P(X=1, Y=1) = \frac{3}{8} \end{aligned}$$

$$\underline{P(W=2)}$$

$$\begin{aligned} P(W=2) &= P(X=0, Y=2) + P(X=1, Y=2) \\ &= \frac{3}{8} \end{aligned}$$

$$P(W=3) = P(X=0, Y=3) + P(X=1, Y=3) \\ = \frac{1}{8}$$

$W$	0	1	2	3
$P(W=w)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(check that the total probability is 1)

# The Expected Value of a Combination of Two Discrete Random Variables

19

If  $W = h(X, Y)$  there are  
two ways to compute  $E(W)$ .

## Proposition

$$E(W) = \sum_{\substack{\text{all } (x, y) \\ \text{possible} \\ \text{values of} \\ (X, Y)}} h(x, y) P_{X, Y}(x, y) \quad (\#)$$

We will illustrate the proposition  
by computing  $E(W)$  for the  
 $W = X + Y$  of pages 12, 13, 14.

In two ways.

First way (without using the proposition)

$W$  is a random variable with pmf given by (b) on page 14 (so we use (b))

$$E(W) = (0)\left(\frac{1}{8}\right) + (1)\left(\frac{2}{8}\right) + (2)\left(\frac{2}{8}\right) + (3)\left(\frac{2}{8}\right) + (4)\left(\frac{1}{8}\right)$$

$$= \frac{2 + 4 + 6 + 4}{8} = \frac{16}{8} = 2$$

Second way (using the proposition)

Now we use (\*) from page 12

$$E(W) = E(X+Y) = \sum_{\text{all } x,y} (x+y) P_{X,Y}(x,y)$$

Sum over the 8 entries of (\*)

$$= (0+0)\left(\frac{1}{8}\right) + (0+1)\left(\frac{2}{8}\right) + (0+2)\left(\frac{1}{8}\right) + (0+3)\left(\frac{0}{8}\right) \\ + (1+0)\left(\frac{0}{8}\right) + (1+1)\left(\frac{1}{8}\right) + (1+2)\left(\frac{2}{8}\right) + (1+3)\left(\frac{1}{8}\right)$$

$$= \frac{2+2}{8} + \frac{2+6+4}{8}$$

$$= \frac{4+12}{8} = 2$$

The first way is easier but we need to compute the pmf of  $W = X + Y$  first. That was hard work, pages 12-14.