Lecture 18

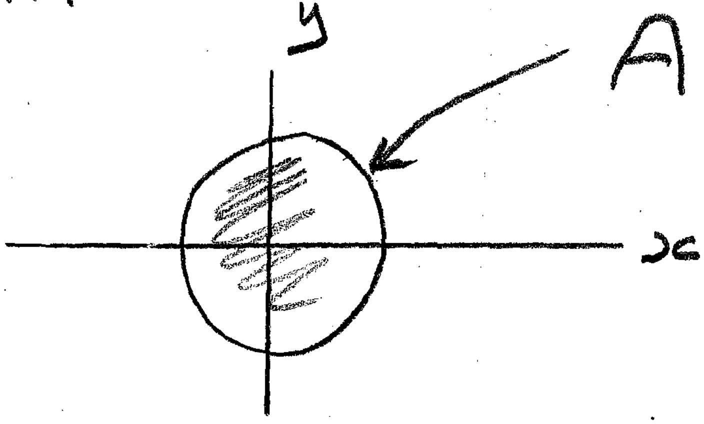
Pairs of Continuous Random Variables

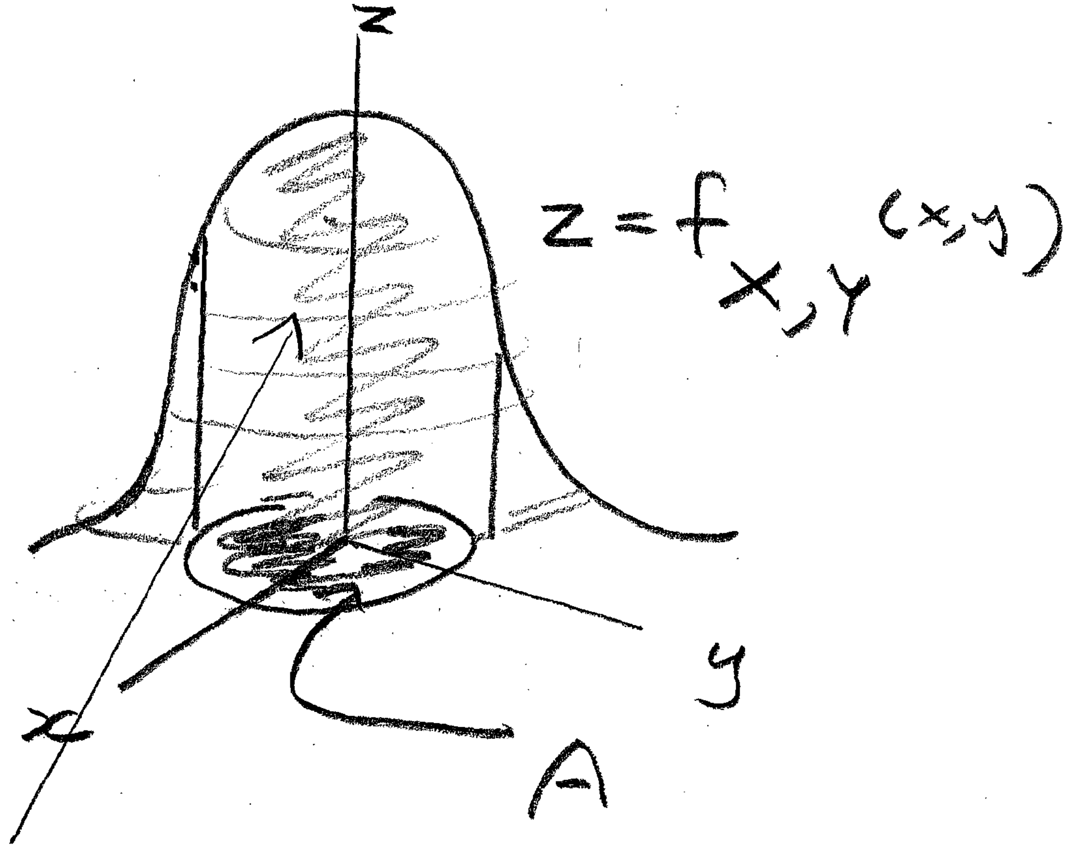
Definition Let X and Y be continuous random variables defined on the same sample space S. Then the joint probability density function, joint pdf, $f_{X,Y}(x,y)$ is the function such that

$$(*) \quad P((X,Y)\in A) = \underbrace{\iint_A f_{X,Y}(x,y)\,dx\,dy}_{\text{double integral}}$$

for any region A in the plane.

Again the geometric interpretation of (*) is very important





$P((X,Y) \in A)$ = the volume under the graph of f and above the region A

For $f(x,y)$ to be a joint pdf for some pair of random variables X and Y it is necessary and sufficient that

$$f(x,y) \geq 0, \text{ all } x, y.$$

and $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

or geometrically, the total volume under the graph of f has to be 1.

Example 5.3 (from text)

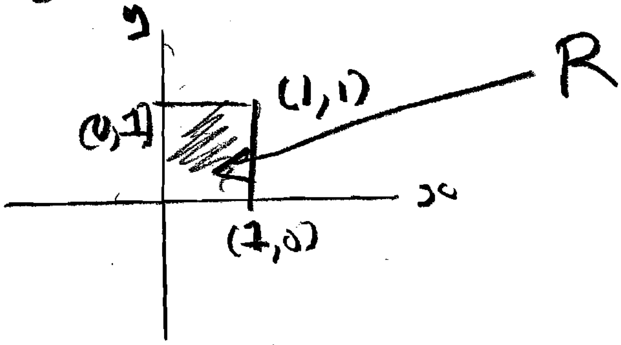
A bank operates a drive-up window and a walkup window. On a randomly selected day, let

X = proportion of time the drive-up facility is in use.

Y = proportion of time the walk-up facility is in use.

The set of possible outcomes for the pair (X, Y) is the square

$$R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$$



Suppose ~~the joint~~ pdf of (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{6}{5}(x + y^2), & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that neither facilty is in use more than $\frac{1}{4}$ of the time.

<u>Solution</u> Neither facilty is in use more than $\frac{1}{4}$ of the time when re-expressed in terms of X and Y is

$X \le \frac{1}{4}$ (the drive-up facilty is in use $\le \frac{1}{4}$ of the time)

and

$Y \le \frac{1}{4}$ (the walk-up facilty is in use $\le \frac{1}{4}$ of the time)

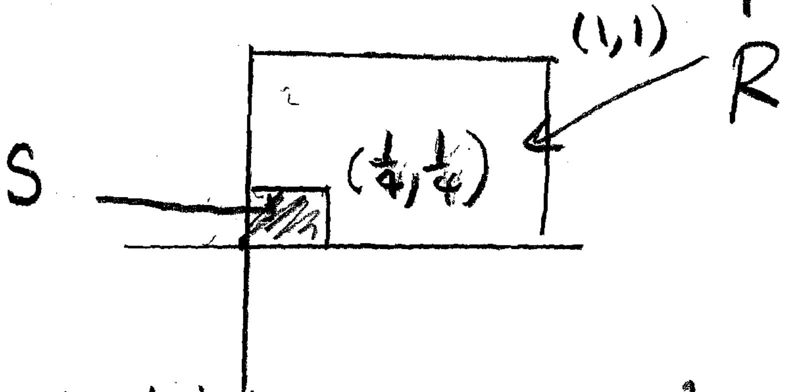
The author has formulated the problem in a confusing fashion, don't worry about it.

So we want

$$P\left(0 \le X \le \tfrac{1}{4},\ 0 \le Y \le \tfrac{1}{4}\right)$$

or $P((X,Y) \in S)$

where S is the small square



This probability is given by

$$\int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= \iint_S \frac{6}{5}(x+y^2)\,dx\,dy \qquad (\#)$$

Remark

For general (X, Y) we have

$$P(a \le X \le b,\ c \le Y \le d) = \int_a^b \int_c^d f_{X,Y}(x,y)\, dx\, dy.$$

Let's do the integral (#). We will do the x-integration first. so

$$P(0 \le X \le \tfrac{1}{4},\ 0 \le Y \le \tfrac{1}{4})$$

$$= \int_0^{\frac{1}{4}} \left(\int_0^{\frac{1}{4}} \frac{6}{5}(x+y^2)\, dx \right) dy$$

$$= \frac{6}{5} \int_0^{\frac{1}{4}} \left(\frac{x^2}{2} + xy^2 \right) \Bigg|_{x=0}^{x=\frac{1}{4}} dy$$

$$= \frac{6}{5} \int_0^{\frac{1}{4}} \left(\frac{1}{32} + \frac{y^2}{4} \right) dy$$

$$= \frac{6}{5} \left[\left(\frac{y}{32} + \frac{y^3}{12} \right) \Big|_{y=0}^{y=\frac{1}{4}} \right]$$

$$= \frac{6}{5} \left[\frac{1}{128} + \frac{1}{(64)(12)} \right]$$

$$= \left(\frac{6}{5}\right)\left(\frac{1}{64}\right)\left(\frac{1}{2} + \frac{1}{12}\right)$$

$$= \left(\frac{6}{5}\right)\left(\frac{1}{64}\right)\left(\frac{7}{\cancel{12}\,2}\right)$$

$$= \frac{7}{640}$$

An exercise in the forgotten art of fractions — more of the same later.

More Theory

Marginal Distributions in the Continuous Case

Problem

Suppose you know the joint pdf $f_{X,Y}(x,y)$ of (X,Y). How do you find the individual pdf's $f_X(x)$ of X and $f_Y(y)$. The answer is

Proposition

(i) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

$(*)$

(ii) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

The formula (*) ~~is~~ the continuous analogue of the formula for the discrete case.

Namely

Discrete Case

$$f_X(x) = \sum_{\text{all } y} f_{X,Y}(y)$$

Continuous Case

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

In the first case we sum away the "extra variable" y and in the second case we integrate it away.

By analogy once again we call $f_X(x)$ and $f_Y(y)$ (obtained via (*)) the marginal densities or marginal pdf's.

Note the $f_X(x)$ and $f_Y(y)$ are the two partial definite integrals of $f_{X,Y}(x,y)$ — see Lecture 16.

<u>Example 5.4</u>

We compute the two marginal pdf's for the bank problem, Example 5.3.

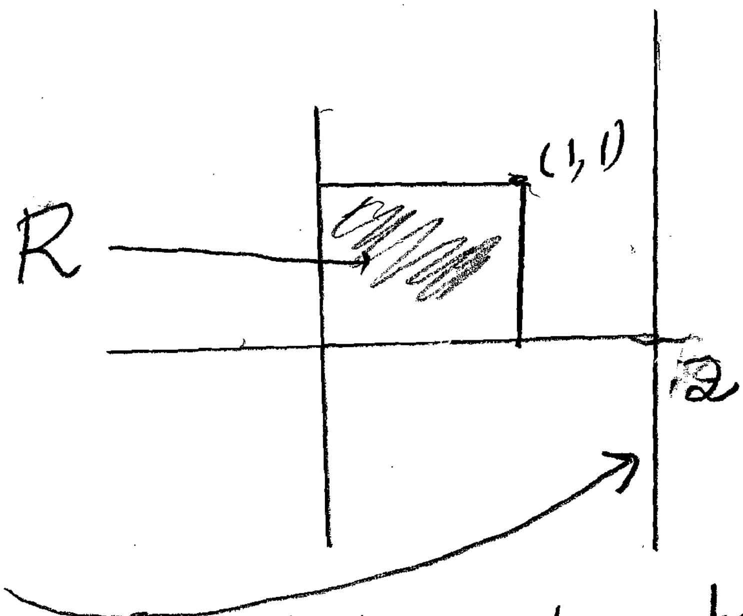
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \begin{cases} \int_0^1 \frac{6}{5}(x+y^2)\,dy & , \quad 0 \le x \le 1 \\ 0 & , \text{ otherwise} \end{cases}$$

this is a little tricky.

The formula for $f_X(x)$ says you integrate $f_{X,Y}(x,y)$ over the vertical

line passing through x.
If x does not satisfy $0 \le x \le 1$ then the vertical line does not pass through the square R where $f_{X,Y}(x,y)$ is non zero



You get $f_X(2)$ by integrating over the line $x = 2$ above which $f_{X,Y}(x,y) = 0$.
Equivalently (without geometry)

$$f_X(2) = \int_{-\infty}^{\infty} f_{X,Y}(2,y)\,dy = \int_{-\infty}^{\infty} 0\,dy = 0$$

Now we finish the job

$$\int_0^1 \frac{6}{5}(x+y^2)\,dy = \frac{6}{5}\int_0^1 (x+y^2)\,dy$$

$$= \frac{6}{5}\left(xy+\frac{y^3}{3}\right)\Big|_{y=0}^{y=1} = \frac{6}{5}\left(x+\frac{1}{3}\right)$$

Similarly

$$f_Y(y) = \begin{cases} \frac{6}{5}\int_0^1 \frac{6}{5}(x+y^2)\,dx, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{6}{5}y^2+\frac{3}{5}, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Independence of Two Continuous Random Variables

Definition

Two continuous random variables X and Y are independent if their joint pdf $f_{X,Y}(x,y)$ is the product of the two marginal pdfs $f_X(x)$ and $f_Y(y)$ so

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

This not true for the bank example pg 5

$$f_{X,Y}(x,y) = \tfrac{6}{5}(x+y^2)$$

not a product

Covariance and Correlation of Pairs of Continuous Random Variables

We continue with a pair of continuous random variables X and Y as before. Again we define

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

and $$\rho_{X,Y} = Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

BUT now

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

We will now compute the $Cov(X,Y)$ and $Corr(X,Y)$ for the book problem. So

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} \frac{6}{5}(x+\frac{1}{3}), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{6}{5}y^2 + \frac{3}{5}, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Let's first do the calculations for X and Y – we need $E(X)$, $E(Y)$, $\sigma_X = \sqrt{V(X)}$ and $\sigma_Y = \sqrt{V(Y)}$

$$E(X) = \int_0^1 x \, \frac{6}{5}\left(x+\frac{1}{3}\right) dx$$

$$= \frac{6}{5}\int_0^1 \left(x^2+\frac{x}{3}\right) dx = \frac{6}{5}\left(\frac{x^3}{3}+\frac{x^2}{6}\right)\Bigg|_{x=0}^{x=1}$$

$$= \frac{6}{5}\left(\frac{1}{3}+\frac{1}{6}\right) = \frac{6}{5}\left(\frac{3}{6}\right) = \frac{3}{5}$$

$$E(X^2) = \int_0^1 x^2 \, \frac{6}{5}\left(x+\frac{1}{3}\right) dx$$

$$= \frac{6}{5}\int_0^1 \left(x^3+\frac{x^2}{3}\right) dx = \frac{6}{5}\left(\frac{x^4}{4}+\frac{x^3}{9}\right)\Bigg|_{x=0}^{x=1}$$

$$= \frac{6}{5}\left(\frac{1}{4}+\frac{1}{9}\right) = \frac{6}{5}\left(\frac{13}{36}\right) = \frac{13}{30}$$

$$V(X) = \frac{13}{30} - \left(\frac{3}{5}\right)^2 = \frac{13}{30} - \frac{9}{25} = \frac{65-54}{150} = \frac{11}{150}$$

$$\sigma_X = \sqrt{\frac{11}{150}} = \frac{1}{5}\sqrt{\frac{11}{6}}$$

$$E(Y) = \int_0^1 y\left(\frac{6}{5}y^2 + \frac{3}{5}\right)dy$$

$$= \frac{6}{5}\int_0^1 y^3\,dy + \frac{3}{5}\int_0^1 y\,dy$$

$$= \left(\frac{6}{5}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{5}\right)\left(\frac{1}{2}\right) = \frac{6}{20} + \frac{3}{10} = \frac{12}{20}$$

$$E(Y^2) = \int_0^1 y^2\left(\frac{6}{5}y^2 + \frac{3}{5}\right)dy$$

$$= \frac{6}{5}\int_0^1 y^4\,dy + \frac{3}{5}\int_0^1 y^2\,dy$$

$$= \left(\frac{6}{5}\right)\left(\frac{1}{5}\right) + \left(\frac{\cancel{3}}{5}\right)\left(\frac{1}{\cancel{3}}\right) = \frac{6}{25} + \frac{1}{5} = \frac{11}{25}$$

$$V(Y) = \frac{11}{25} - \frac{144}{400} = \frac{176}{400} - \frac{144}{400} = \frac{32}{400} = \frac{2}{25}$$

$$\sigma_Y = \sqrt{\frac{2}{25}} = \frac{1}{5}\sqrt{2}$$

Finally we need

$$E(XY) = \int_0^1 \int_0^1 (xy)\frac{6}{5}(x+y^2)\,dx\,dy$$

$$= \int_0^1 \int_0^1 \underbrace{xy\,\frac{6}{5}x}_{\text{product function}}\,dx\,dy + \int_0^1 \int_0^1 \underbrace{xy\,\frac{6}{5}y^2}_{\text{product function}}\,dx\,dy$$

$$= \frac{6}{5}\left(\int_0^1 x^2\,dx\right)\left(\int_0^1 y\,dy\right) + \frac{6}{5}\left(\int_0^1 x\,dx\right)\left(\int_0^1 y^3\,dy\right)$$

$$= \left(\frac{6}{5}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{6}{5}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$$

$$= \left(\frac{6}{5}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}+\frac{1}{4}\right) = \left(\frac{\cancel{6}}{5}\right)\left(\frac{1}{2}\right)\left(\frac{7}{\cancel{12}_{\,2}}\right) = \frac{7}{20}$$

Now we can reap the fruits of our labours.

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{7}{20} - \left(\frac{3}{5}\right)\left(\frac{12}{20}\right)$$

$$= \frac{7}{20} - \frac{36}{100} = \frac{35}{100} - \frac{36}{100}$$

$$Cov(X,Y) = \frac{-1}{100}$$

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\frac{-1}{100}}{\left(\frac{1}{5}\sqrt{\frac{11}{6}}\right)\left(\frac{1}{5}\sqrt{2}\right)}$$

$$= \left(\frac{-1}{100}\right)\left(\frac{5}{\sqrt{\frac{11}{\cancel{6}_{3}}}}\right)\left(\frac{5}{\sqrt{\cancel{2}}}\right) = -\frac{1}{4}\left(\frac{1}{\sqrt{\frac{11}{3}}}\right) = -\frac{\sqrt{3}}{4\sqrt{11}}$$

Independence of Continuous Random Variables

Definition

Two continuous random variables X and Y are independent if the joint pdf is the product of the two marginal pdf's

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

(so $\Leftrightarrow$ the joint pdf is a product function)

So in Example 5.3, page 4, X and Y are NOT independent.