Lecture 19

More Than Two Random Variables

Definition

If $X_1, X_2, \ldots, X_n$ are discrete random variables defined on the same sample space then their joint pmf is the function

$$p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n$$

If $X_1, X_2, \ldots, X_n$ are continuous then their joint pdf is the function $f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$ such that for any region A in $\mathbb{R}^n$

$$P((X_1, X_2, \dots, X_n) \in A) = \underbrace{\int \cdots \int_A f_{X_1, X_2, \dots, X_n}(x_1, \dots, x_n)\, dx_1 \cdots dx_n}_{\text{n-fold multiple integral}}$$

Definition

The discrete random variables $X_1, X_2, \dots, X_n$ are independent if

$$p_{X_1, \dots, X_n}(x_1, \dots, x_n) = p_{X_1}(x_1)\, p_{X_2}(x_2) \cdots p_{X_n}(x_n).$$

Equivalently

$$P(X_1 = x_1, \dots, X_n = x_n) = P(X_1 = x_1) \cdots P(X_n = x_n)$$

The continuous random variables $X_1, X_2, \dots, X_n$ are independent if

$$f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = f_{X_1}(x_1)\, f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

<u>Definiteon</u>

$X_1, X_2, \dots, X_n$ are <u>pairwise independent</u> if each pair X_i, X_j $(i \neq j)$ is independent.

We will now see

Pairwise independence of random variables $\not\Longrightarrow$ Independence of random variables

$\Longleftarrow$

First we will prove $\Leftarrow$

Theorem

$X_1, X_2, \ldots, X_n$ independent

$\Longrightarrow X_1, X_2, \ldots, X_n$ are pairwise independent.

From now on we will restrict to the case $n=3$ so we have THREE random variables X, Y, Z

How do we get

$P_{X,Y}(x,y)$ from $P_{X,Y,Z}(x,y,z)$

Answer (left to you to prove)

$$P_{X,Y}(x,y) = \sum_{\text{all } z} P_{X,Y,Z}(x,y,z) \qquad (\#)$$

Now we can prove

X, Y, Z independent

$\Longrightarrow X, Y$ independent

Since X, Y, Z are independent we have

$$P_{X,Y,Z}(x,y,z) = P_X(x)\, P_Y(y)\, P_Z(z) \qquad (\#\#)$$

Now plug the RHS of (##) into the RHS of (#)

$$P_{X,Y}(x,y) = \sum_{\text{all } z} P_X(x)\, P_Y(y)\, P_Z(z)$$

$$= P_X(x)\, P_Y(y) \underbrace{\sum_{\text{all } z} P_Z(z)}_{=1}$$

$$= P_X(x)\, P_Y(y)$$

This proves X and Y are independent. Identical proofs prove the pairs X, Z and Y, Z are independent.

Now we construct X, Y, Z (actually X_A, X_B, X_C) so that each pair is independent but the triple X, Y, Z is not independent

A Variation on the Cool Counterexample

Lets go back to the "cool counterexample", Lecture 16, page 18 of three events A, B, C which are pairwise independent but no independent so

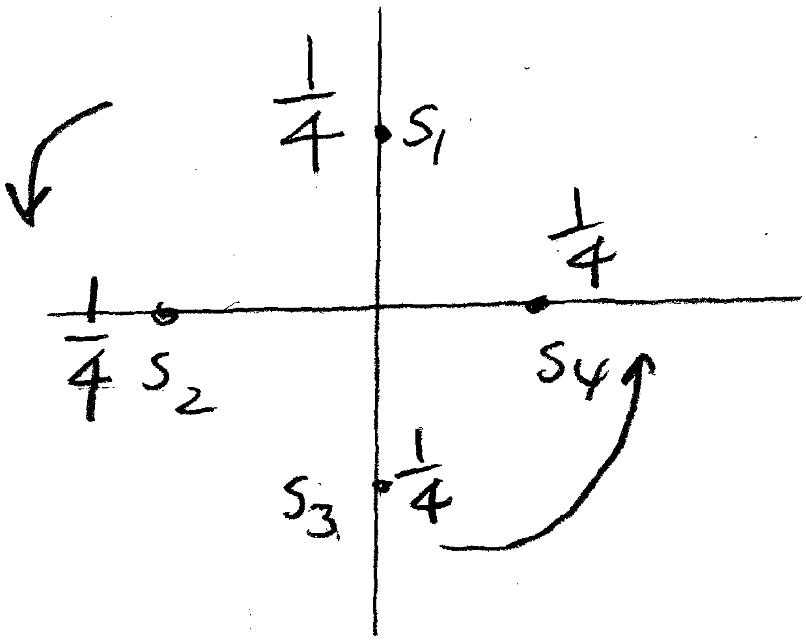
$$P(A \cap B \cap C) \neq P(A)P(B)P(C).$$

The idea is to convert the three events to random variables X_A, X_B, X_C so that

$X_A = 1$ on A and 0 on A' etc.

In fact we won't need the corner points $(-1,-1)$, $(-1,1)$, $(1,-1)$ and $(1,1)$ we put $s_1 = (0,1)$, $s_2 = (-1,0)$, $s_3 = (0,1)$, $s_4 = (1,0)$ and retain their probabilities so

$$P(\{s_j\}) = \frac{1}{4}, \quad 1 \le j \le 4$$

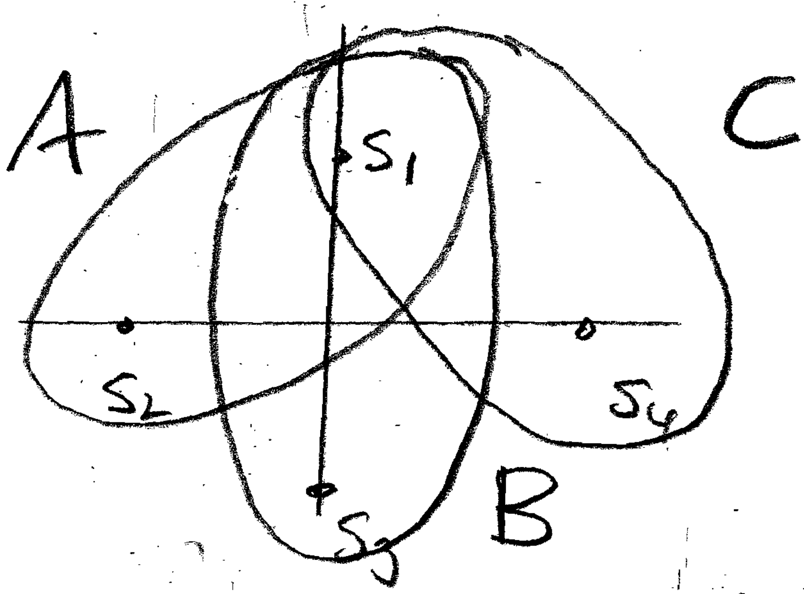


We define

$$A = \{s_1, s_2\}$$

$$B = \{s_1, s_3\}$$

$$C = \{s_1, s_4\}$$



We define χ_A, χ_B, χ_C on S by

$$\chi_A(s_j) = \begin{cases} 1, & \text{if } s_j \in A \\ 0, & \text{if } s_j \notin A \end{cases}$$

$$X_B(s_j) = \begin{cases} 1, & \text{if } s_j \in B \\ 0, & \text{if } s_j \notin B \end{cases}$$

$$X_C(s_j) = \begin{cases} 1, & \text{if } s_j \in C \\ 0, & \text{if } s_j \notin C \end{cases}$$

So $P(X_A = 1) = P(\{s_1, s_2\}) = \frac{1}{2}$

$P(X_A = 0) = P(\{s_3, s_4\}) = \frac{1}{2}$

and similarly for X_B and X_C.

So X_A, X_B and X_C

are Bernoulli random variables

Let's compute the joint pmf of X_A and X_B. We know the margin

$X_A \backslash X_B$	0	1	
0			$\frac{1}{2}$
1			$\frac{1}{2}$
	$\frac{1}{2}$	$\frac{1}{2}$	

The subset where $X_A = 1$ is the subset $\{s_1, s_2\}$ so we write on equality of events

$$(X_A = 1) = \{s_1, s_2\}$$

similarly

$$(X_A = 0) = \{s_3, s_4\}$$

$$(X_B = 1) = \{s_1, s_3\}, \quad (X_B = 0) = \{s_2, s_4\}$$

$$(X_C = 1) = \{s_1, s_4\}, \quad (X_C = 0) = \{s_2, s_3\}$$

Hence

$$(X_A=0)\cap(X_B=0)=\{s_4\}$$

so $P(X_A=0, X_B=0)=\frac{1}{4}$

$$(X_A=0)\cap(X_B=1)=\{s_3\}$$

so $P(X_A=0, X_B=1)=\frac{1}{4}$

$$(X_A=1)\cap(X_B=0)=\{s_2\}$$

$$P(X_A=1, X_B=0)=\frac{1}{4}$$

$$(X_A=1)\cap(X_B=1)=\{s_1\}$$

$$P(X_A=1, X_B=1)=P(\{s_1\})=\frac{1}{4}$$

etc

So the joint pmf of X_A and X_B is

$X_A \backslash X_B$	0	1	
0	1/4	1/4	$\frac{1}{2}$
1	1/4	$\frac{1}{4}$	$\frac{1}{2}$
	$\frac{1}{2}$	$\frac{1}{2}$	

so X_A and X_B are independent.

The same is true for X_A and X_C and X_B and X_C.

Now we show the triple X_A, X_B and X_C is NOT independent.

We will show

$$P(X_A = 1, X_B = 1, X_C = 1)$$

$$\neq P(X_A = 1)\, P(X_B = 1)\, P(X_C = 1)$$

The RHS $= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$

The left-hand side is the probability of the event

$$(X_A = 1) \cap (X_B = 1) \cap (X_C = 1)$$

$$= \{s_1, s_2\} \cap \{s_1, s_3\} \cap \{s_1, s_4\}$$

$$= \{s_1\}.$$

So

$$P(X_A=1, X_B=1, X_C=1) = P(\{s_1\}) = \frac{1}{4}$$

so

$$LHS = \frac{1}{4}$$

$$RHS = \frac{1}{8}$$

Remark

This counterexample is more or less the same as the "cool counterexample". We just replaced (more or less) A, B, C by their "characteristic functions".