Stat 400

Practice Final Solutions

(a) $E(X) = 2$

(b) $E(X^2) = \frac{14}{3}$

(c) $V(X) = \frac{14}{3} - (2)^2 = \frac{2}{3}$

(d)
$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{3}, & 1 \le x < 2 \\ \frac{2}{3}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

(e) $P(Y=0) = P(X=1) = \frac{1}{3}$

$P(Y=1) = P(X=2) = \frac{1}{3}$

$P(Y=2) = P(X=3) = \frac{1}{3}$

2(a) $E(X) = \frac{1}{3}$

(b) $E(X^2) = \frac{1}{6}$

so $V(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$

(c) $F(x) = \begin{cases} 0, & x < 0 \\ 2(x - x^2/2), & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$

(d) Note first the median $\tilde{\mu}$ must satisfy

$$0 < \tilde{\mu} < 1 \qquad (*)$$

Now also $F(\tilde{\mu}) = \frac{1}{2}$ so $\tilde{\mu}$ is a root of

$$2(x - x^2/2) = \frac{1}{2}$$

$$x^2 - 2x + \frac{1}{2} = 0$$

$$x = \frac{2 \pm \sqrt{4 - (4)(\frac{1}{2})}}{2} = 1 \pm \frac{\sqrt{2}}{2}$$

Since $\tilde{\mu}$ satisfies $(*)$ and $1 + \frac{\sqrt{2}}{2} > 1$ we must have $\tilde{\mu} = 1 - \frac{\sqrt{2}}{2}$

(e) $\eta_{.75}$ satisfies $2(x - x^2/2) = \frac{3}{4}$ and $0 < \eta_{.75} \leq 1$

$x = 1 \pm \frac{1}{2}$ so $\eta_{.75} = \frac{1}{2}$

3. Let A = Jack and Jill sit together 3

B = Dick and Jane sit "

We want $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

There are $4! = 24$ ways of seating four people in adjacent seats.

$\#(A) =$ JJ D J "glue Jack and Jill together"

$= 2(3!) = (2)(6) = 12$

$\#(B) = 12$ same argument

$\#(A \cap B) =$ JJ DJ

glue Jack and Jill

glue Dick and Jane

$= 2(2)(2) = 8$

$\#(A \cup B) = 12 + 12 - 8 = 16$

$$P(A \cup B) = \frac{16}{24} = \frac{2}{3}$$

#4 (a)

X\Y	0	1	
0	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
	$\frac{1}{4}$	$\frac{3}{4}$	Y \ X

X	0	1
p(X)	$\frac{1}{4}$	$\frac{3}{4}$

Y	0	1
p(Y)	$\frac{1}{4}$	$\frac{3}{4}$

(b) No. $P(X=0, Y=0) = 0 \neq \frac{1}{16} = P(X=0) \cdot P(Y=0)$

(c) $Z = X+Y$

$Z=0 \Leftrightarrow (X=0, Y=0) \quad P(Z=0)=0$

$Z=1 \Leftrightarrow \begin{cases} (X=1, Y=0) \\ (X=0, Y=1) \end{cases} \quad \underline{P(Z=1)} = \frac{1}{2}$

$\underline{Z=2} \Rightarrow \underline{\underline{P(Z=2)}} = \frac{1}{2}$ $\boxed{\text{pmf}}$

(d) $Cov(X,Y)$: $E(XY) = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$

$E(X) = \frac{3}{4} \qquad E(Y) = \frac{3}{4}$

$\Rightarrow Cov(X,Y) = E(XY) - E(X) \cdot E(Y) = \frac{1}{2} - \frac{9}{16} = -\frac{1}{16}$

(e) $$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$

$Var(X) = E(X^2) - E(X) = \frac{3}{4} - (\frac{3}{4})^2 = \frac{3}{16}$

$$\Rightarrow \rho_{XY} = \frac{-\frac{1}{16}}{\frac{3}{16}} = -\frac{1}{3}$$

5 (ii) Lower-tailed confidence interval ($\underline{\sigma^2 \text{ unknown}}$) is

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$$\left(-\infty,\ \bar{x} + t_{\alpha,n-1} \frac{s}{\sqrt{26}}\right)$$

$$\left(-\infty,\ 24.36 + t_{.05,25} \frac{370.69}{5.1}\right)$$

$$= \left(-\infty,\ 24.36 + (1.708) \frac{370.69}{5.1}\right)$$

$$= \left(-\infty,\ 24.63 + 124.145\right)$$

$$= \left(-\infty,\ 148.775\right)$$

(iii) lower-tailed prediction interval (σ^2 unknown)

$$\left(-\infty,\ \bar{x} + t_{\alpha,n-1} \sqrt{\frac{n+1}{n}}\, s\right)$$

$$= \left(-\infty,\ 24.36 + 1.708 \sqrt{\frac{27}{26}}\ 370.69\right)$$

$$= \left(-\infty,\ 24.36 + (1.708)(1.019)(370.69)\right)$$

$$= \left(-\infty,\ 24.36 + 645.168\right)$$

$$= \left(-\infty,\ 669.528\right)$$

6. **Theorem A** $\quad T = \dfrac{\overline{X} - \mu}{(S/\sqrt{n})} \sim t_{n-1}$ 6

We want to prove

$$P\left(\mu \in \left(\overline{X} - t_{\alpha, n-1} \frac{S}{\sqrt{n}}, \infty\right)\right) = 1 - \alpha$$

$$\text{LHS} = P\left(\overline{X} - t_{\alpha, n-1} \frac{S}{\sqrt{n}} < \mu\right)$$

SWAP

$$= P\left(\overline{X} - \mu < t_{\alpha, n-1} \frac{S}{\sqrt{n}}\right)$$

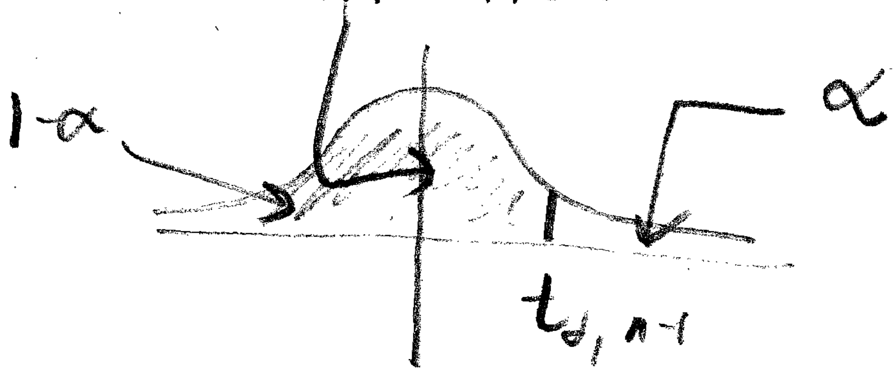
DIVIDE

$$= P\left(\frac{\overline{X} - \mu}{S/\sqrt{n}} < t_{\alpha, n-1}\right)$$

By Theorem A

$$= P(T < t_{\alpha, n-1})$$

= this area



= $1 - \alpha$ from the picture