Stat 400 Final Spring 2010

Solutions

1. (a) $$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x(n+1)x^n\,dx$$

$$= (n+1)\int_0^1 x^{n+1}\,dx = \frac{n+1}{n+2}$$

(b) $$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2(n+1)x^n\,dx$$

$$= (n+1)\int_0^1 x^{n+2}\,dx = \frac{n+1}{n+3}$$

(c) $$V(X) = E(X^2) - E(X)^2 = \frac{n+1}{n+3} - \left(\frac{n+1}{n+2}\right)^2$$

$$= \frac{(n+1)(n+2)^2 - (n+1)^2}{(n+3)(n+2)^2} = \frac{(n+1)[n^2+4n+4-n-1]}{(n+3)(n+2)^2}$$

$$= \frac{(n+1)(n^2+3n+3)}{(n+3)(n+2)^2}$$

(d)

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ x^{n+1} & , \quad 0 \leq x \end{cases}$$

(e) $\tilde{\mu}$ satisfies $F(\tilde{\mu}) = \frac{1}{2}$

$$\Leftrightarrow (\tilde{\mu})^{n+1} = \frac{1}{2} \Leftrightarrow \tilde{\mu} = \frac{1}{\sqrt[n+1]{2}}$$

(f) $\eta(.9) = 90^{th}$ percentile satisfies

$$F(\eta(.9)) = .9 \Leftrightarrow \eta(.9)^{n+1} = \frac{9}{10}$$

$$\Leftrightarrow \eta(.9) = \left(\frac{9}{10}\right)^{\frac{1}{n+1}}$$

90th percentile

2(a) Let X = # of passengers who get off at the first stop

Whether or not a randomly selected passenger gets off at the first stop is a Bernoulli trial (coin flip) with success probability $\frac{1}{3}$ – that is $X \sim Bin(n, \frac{1}{3})$ so

$$P(X=k) = \binom{n}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$$

In particular the probability that nobody gets off at the first stop is

(*) $$P(X=0) = \left(\frac{2}{3}\right)^n \quad (n = \text{\# of passengers})$$

(b) Let A = no one gets off $= (X=0)$.

B = it is Sunday

so B' = it is not Sunday

Hence from (*) we have

(**) $$\begin{cases} P(X=0 \mid B) = \left(\frac{2}{3}\right)^2 & (\text{because } n=2) \\ P(X=0 \mid B') = \left(\frac{2}{3}\right)^4 & (\text{because } n=4) \end{cases}$$

We want $P(B \mid X=0)$

By Bayes' Theorem

$$P(B \mid X=0) = \frac{P(X=0 \mid B)\,P(B)}{P(X=0 \mid B)\,P(B) + P(X=0 \mid B')\,P(B')}$$

$$= \frac{\left(\frac{2}{3}\right)^2\left(\frac{1}{7}\right)}{\left(\frac{2}{3}\right)^2\left(\frac{1}{7}\right) + \left(\frac{2}{3}\right)^4\left(\frac{6}{7}\right)}$$

factor out $\left(\frac{2}{3}\right)^2\left(\frac{1}{7}\right)$

$$= \frac{1}{1 + \left(\frac{2}{3}\right)^2(6)} = \frac{1}{\frac{9+(4)(6)}{9}}$$

$$= \frac{9}{9+(4)(6)} = \frac{9}{33} = \frac{3}{11}$$

3 (a) $\#(S) = \underbrace{(52)(51)(50)}_{P_{52,3}}$

(since order counts)

(b) $\#(A) = (13)(51)(50)$

$P(A) = \dfrac{\#(A)}{\#(S)} = \dfrac{(13)\cancel{(51)}\cancel{(50)}}{(52)\cancel{(51)}\cancel{(50)}} = \dfrac{13}{52} = \dfrac{1}{4}$

(c) $\#(B) = (51)(50)(13)$

$P(B) = \dfrac{\cancel{(51)}\cancel{(50)}(13)}{(52)\cancel{(51)}\cancel{(50)}} = \dfrac{13}{52} = \dfrac{1}{4}$

(d) $\#(C) = \underbrace{(51)(50)\cdots(3)(2)(1)}_{(51)!}\ 13$

$\#(S) = 52!$

$P(S) = \dfrac{\#(C)}{\#(S)} = \dfrac{(51)!\ (13)}{(52)!}$

$= \dfrac{13}{52} = \dfrac{1}{4}$

4.

X \ Y	-1	0	1
0	a	b	c
1	0	e	0
	a		c

Put zeroes here makes all six terms in the sum for $E(XY)$ equal to zero.

$E(Y) = (-1)(a) + (1)(c) = c - a$

So if we put $a = c$ we get $E(Y) = 0$

so $Cov(X, Y) = E(XY) - E(X)E(Y)$
$= 0 - E(X)0 = 0$

So for

X \ Y	-1	0	1
0	a	b	a
1	0	e	0

We have $Cov(X, Y) = 0$

(b). There is a typo here.

We want $\mathrm{Cov}(X,Y)=0$

but X and Y are <u>dependent</u>.

$x \backslash y$	-1	0	1	
0	a	b	a	$2a+b$
1	0	e	0	e
	a	$b+e$	a	

<u>If</u> $a \neq 0$ then X and Y

are dependent.

But we need $2a+b+e=1$

Take them all to be $\frac{1}{4}$

$X \backslash Y$	-1	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
1	0	$\frac{1}{4}$	0

5. $\hat{p} = \frac{10}{100} = \frac{1}{10}$

$\hat{q} = \frac{9}{10}$

so $\alpha = .10$ so $\alpha/2 = .05$

$z_{\alpha/2} = z_{.05} = 1.645$ from the table

$\frac{\hat{p}\,\hat{q}}{n} = \frac{\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)}{100} = \frac{9}{10^4} = \left(\frac{3}{10^2}\right)^2$

$\sqrt{\frac{\hat{p}\hat{q}}{n}} = \sqrt{\left(\frac{3}{10^2}\right)^2} = \frac{3}{10^2} = .03$

$\left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}},\ \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}\right)$

$= (.1 - (1.645)(.03),\ .1 + (1.645)(.03))$

$= (.1 - .04935,\ .1 + .04935)$

$= (.05065,\ .14935)$

6 (a) We want

$$P\left(\lambda \in \left(\frac{c}{2n\bar{X}}, \infty\right)\right) = 1-\alpha$$

$$LHS = P\left(\frac{c}{2n\bar{X}} < \lambda\right)$$

$$= P(c < 2n\lambda\bar{X})$$

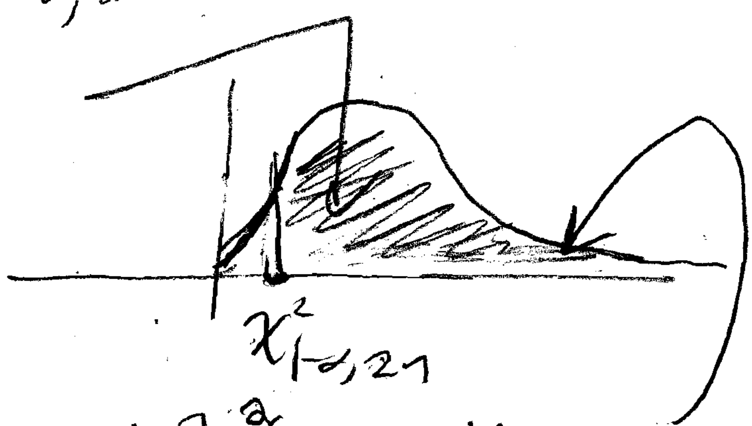
$$= P(c < V)$$

Choose $c = \chi^2_{1-\alpha, 2n}$

then

$$LHS = P(\chi^2_{1-\alpha, 2n} < V)$$

this area



But, by definition of $\chi^2_{1-\alpha, 2n}$, this area is $1-\alpha$
so $LHS = 1-\alpha$.

(b) Repeat (a) with $c = \chi^2_{1-\alpha, 2n}$