Stat 400

Midterm 2 Solutions

1. This is the "cool counterexample", see Lecture 16.

(a)

x	-1	0	1
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

and the same for Y.

(b) No, since if they were independent the upper left entry would be $(\frac{1}{4})(\frac{1}{4}) = \frac{1}{16}$ whereas it is 0.

(c) W takes values $-2, -1, 0, -1, 2$

w	-2	-1	0	1	2
$P(W=w)$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0

(d) $Cov(X,Y) = 0+0+0$
$+0+0+0$
$+0+0+0 = 0$

there are nine terms because $3 \times 3 = 9$

x y

2.

$$L(\theta) = \frac{x_1^{\theta}}{\theta+1} \frac{x_2^{\theta}}{\theta+1} \cdots \frac{x_n^{\theta}}{\theta+1}$$

$$L(\theta) = \frac{(x_1 x_2 \cdots x_n)^{\theta}}{(\theta+1)^n}$$

so

$$h(\theta) = \ln L(\theta) = \theta \ln(x_1 \cdots x_n) - n \ln(\theta+1)$$

$$h'(\theta) = \ln(x_1 x_2 \cdots x_n) - \frac{n}{\theta+1}$$

so

$$h'(\theta) = 0 \iff \frac{n}{\theta+1} = \ln(x_1 x_2 \cdots x_n)$$

$$\frac{\theta+1}{n} = \frac{1}{\ln(x_1 x_2 \cdots x_n)}$$

$$\theta + 1 = \frac{n}{\ln(x_1 x_2 \cdots x_n)}$$

so $\theta = \theta_0 = \dfrac{n}{\ln(x_1 x_2 \cdots x_n)} - 1$

so

$$\hat{\theta}_{mle} = \frac{n}{\ln(X_1 X_2 \cdots X_n)} - 1$$

Extra

You don't need to do this but to be correct we should check $h'' > 0$ at this point

$$h''(\theta) = \frac{n}{(\theta+1)^2}$$

so $h''(\theta) > 0$ everywhere!

so $h(\theta)$ has a max at the point θ_0

so $L(\theta) = e^{h(\theta)}$ also has a max at θ_0

BECAUSE

A positive function $f(x)$ has a maximum at $x_0 \iff g(x) = e^{f(x)}$ has a maximum at x_0. You can replace e^x by any other strictly increasing function.

3. We did this in class and it was on Practice Midterm 2 so this is the ultimate "good citizen's proof".

$$E(\overline{X}) = E\left(\tfrac{1}{n}(X_1 + \cdots + X_n)\right)$$

$$= \tfrac{1}{n} E(X_1 + \cdots + X_n)$$

$$= \tfrac{1}{n}\left(E(X_1) + \cdots + E(X_n)\right)$$

But $E(X_1) = E(X_2) = \cdots E(X_n) = \mu$

so

$$E(\overline{X}) = \tfrac{1}{n}(n\mu) = \mu$$

$\square$

4 (a) If John shoots Dave then Bob shoots next. His only choice is to shoot at John. But Bob never misses so John dies before he can get his second shot off

P(John gets a second shot) = 0

(b) If John shoots Bob then Dave shoots next. He misses John with probability $\frac{1}{2}$ so

P(John gets a second shot) = $\frac{1}{2}$.

(c) If John shoots at the floor Dave shoots next. Dave will shoot at Bob because Bob is more dangerous to him.

There are two cases.

1. Dave hits Bob. Then John shoots (at Dave) next so John gets a second shot

2. Dave misses Bob
Then Bob shoots next.
But Bob shoots at Dave because Dave is more dangerous to him than John. Since Bob never misses he hits Dave.
So John gets a second shot (at Bob).

So using strategy (c)

$P(\text{John gets a second shot}) = 1$

d) Since shooting at the floor <u>guarantees</u> John a second shot and hitting either Bob or Dave does not, shooting at the floor is John's best strategy.

<u>Later</u>

We will show that

$$P(\text{John survives} \mid \text{given (a)}) = 0$$

$$P(\text{John survives} \mid \text{given (b)}) = \frac{3}{13}$$

$$P(\text{John survives} \mid \text{given (c)}) = \frac{3}{10}$$

It's not so easy.