

Practice Final

Solutions

1 (a) $E(X) = 2$

(b) $E(X^2) = \frac{14}{3}$

(c) $V(X) = \frac{14}{3} - (2)^2 = \frac{2}{3}$

(d)
$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{3}, & 1 \leq x < 2 \\ \frac{2}{3}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

(e) $P(Y=0) = P(X=1) = \frac{1}{3}$

$$P(Y=1) = P(X=2) = \frac{1}{3}$$

$$P(Y=2) = P(X=3) = \frac{1}{3}$$

$$2(a) \quad E(X) = \frac{1}{3}$$

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$$(b) \quad E(X^2) = \frac{1}{6}$$

$$\text{so } V(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

$$(c) \quad F(x) = \begin{cases} 0, & x < 0 \\ 2(x - \frac{x^2}{2}), & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

(d) Note first the median $\tilde{\mu}$ must satisfy

$$0 < \tilde{\mu} < 1 \quad (*)$$

Now also $F(\tilde{\mu}) = \frac{1}{2}$ so $\tilde{\mu}$ is a root of

$$2(x - \frac{x^2}{2}) = \frac{1}{2}$$

$$x^2 - 2x + \frac{1}{2} = 0$$

$$x = \frac{2 \pm \sqrt{4 - (4)(\frac{1}{2})}}{2} = 1 \pm \frac{\sqrt{2}}{2}$$

Since $\tilde{\mu}$ satisfies (*) and $1 + \frac{\sqrt{2}}{2} > 1$

we must have $\tilde{\mu} = 1 - \frac{\sqrt{2}}{2}$

(e) $\eta_{.75}$ satisfies $2(x - \frac{x^2}{2}) = \frac{3}{4}$ and $0 < \eta_{.75} < 1$
 $x = 1 \pm \frac{1}{2}$ so $\eta_{.75} = \frac{1}{2}$

3. Let $A =$ Jack and Jill sit together ³
 $B =$ Dick and Jane sit "

We want $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

There are $4! = 24$ ways of seating four people in adjacent seats.

$$\begin{aligned} \#(A) &= \begin{array}{c} \text{D J} \\ \text{JJ} \end{array} \\ &\quad \uparrow \uparrow \uparrow \text{ "glue Jack and Jill together" } \\ &= 2(3!) = (2)(6) = 12 \end{aligned}$$

$$\#(B) = 12 \quad \text{same argument glue Jack and Jill}$$

$$\begin{aligned} \#(A \cap B) &= \begin{array}{c} \text{JJ} \quad \text{DJ} \\ \text{JJ} \quad \text{DJ} \end{array} \\ &\quad \uparrow \uparrow \text{ "glue Dick and Jane"} \\ &= 2(2)(2) = 8 \end{aligned}$$

$$\#(A \cup B) = 12 + 12 - 8 = 16$$

$$P(A \cup B) = \frac{16}{24} = \frac{2}{3}$$

4 (a)

X \ Y	0	1	
0	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
	$\frac{1}{4}$	$\frac{3}{4}$	X

X	0	1
P(X)	$\frac{1}{4}$	$\frac{3}{4}$
Y	0	1
P(Y)	$\frac{1}{4}$	$\frac{3}{4}$

b) No. $P(X=0, Y=0) = 0 \neq \frac{1}{16} = P(X=0) \cdot P(Y=0)$

(c) $Z = X + Y$

$Z=0 \Leftrightarrow (X=0, Y=0) \quad P(Z=0) = 0$

$Z=1 \Leftrightarrow \begin{cases} (X=1, Y=0) \\ (X=0, Y=1) \end{cases} \quad P(Z=1) = \frac{1}{2}$

$Z=2 \Rightarrow P(Z=2) = \frac{1}{2}$ pmf

(d) $Cov(X, Y) = E(XY) = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$

$E(X) = \frac{3}{4} \quad E(Y) = \frac{3}{4}$

$\Rightarrow Cov(X, Y) = E(XY) - E(X) \cdot E(Y) = \frac{1}{2} - \frac{9}{16} = -\frac{1}{16}$

(e) $\rho_{XY} = \frac{Cov(X, Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$

$Var(X) = E(X^2) - E(X)^2 = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{16}$

$\Rightarrow \rho_{XY} = \frac{-\frac{1}{16}}{\frac{3}{16}} = -\frac{1}{3}$

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(i) Lower-tailed confidence interval (σ^2 unknown) is

$$(-\infty, \bar{x} + t_{\alpha, n-1} \frac{s}{\sqrt{26}})$$

$$(-\infty, 24.36 + t_{.05, 25} \frac{370.69}{5.1})$$

$$= (-\infty, 24.36 + (1.708) \frac{370.69}{5.1})$$

$$= (-\infty, 24.36 + 124.145)$$

$$= (-\infty, 148.775)$$

(ii) Lower-tailed prediction interval (σ^2 unknown)

$$(-\infty, \bar{x} + t_{\alpha, n-1} \sqrt{\frac{n+1}{n}} s)$$

$$= (-\infty, 24.36 + 1.708 \sqrt{\frac{27}{26}} 370.69)$$

$$= (-\infty, 24.36 + (1.708)(1.019)(370.69))$$

$$= (-\infty, 24.36 + 645.168)$$

$$= (-\infty, 669.528)$$

6. Theorem A $T = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim t_{n-1}$ 6

We want to prove

$$P(\mu \in (\bar{X} - t_{\alpha, n-1} \frac{S}{\sqrt{n}}, \bar{X} + t_{\alpha, n-1} \frac{S}{\sqrt{n}})) = 1 - \alpha$$

$$\text{LHS} = P(\bar{X} - t_{\alpha, n-1} \frac{S}{\sqrt{n}} < \mu)$$

SWAP

$$= P(\bar{X} - \mu < t_{\alpha, n-1} \frac{S}{\sqrt{n}})$$

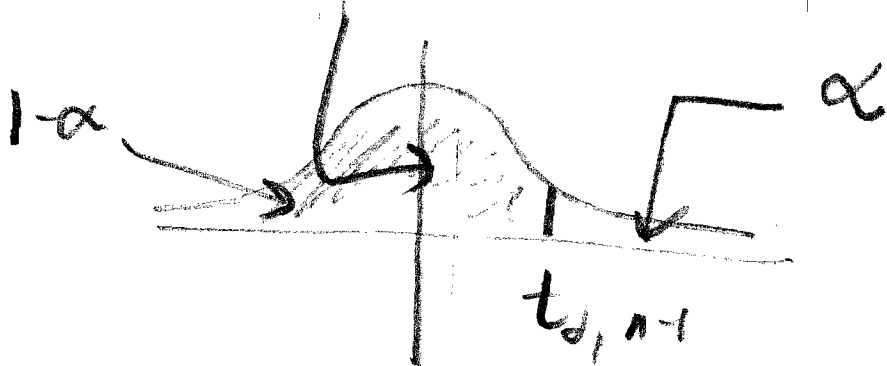
DIVIDE

$$= P\left(\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} < t_{\alpha, n-1}\right)$$

By Theorem A

$$= P(T < t_{\alpha, n-1})$$

= this area



= $1 - \alpha$ from the picture