The confidence interval formulas for the mean in an normal distribution when σ is unknown

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1 Introduction

In this lecture we will derive the formulas for the symmetric two-sided confidence interval and the lower-tailed confidence intervals for the mean in a normal distribution *when the variance σ^2 is unknown.* At the end of the lecture I assign the problem of proving the formula for the upper-tailed confidence interval . We will need the following theorem from probability theory. Recall that $\overline{X}$ is the sample mean (the point estimator for the populations mean μ) and S^2 is the sample variance, the point estimator for the unknown population variance σ^2.
We will need the following theorem from Probability Theory.

Theorem 1. $(\overline{X} - \mu)/\frac{S}{\sqrt{n}}$ *has t-distribution with $n-1$ degrees of freedom.*

2 The two-sided confidence interval formula

Now we can prove the theorem from statistics giving the required confidence interval for μ. Note that it is symmetric around $\overline{X}$. There are also asymmetric two-sided confidence intervals. We will discuss them later. This is one of the basic theorems that you have to learn how to prove.

Theorem 2. *The random interval $T = (\overline{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\ ,\ \overline{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}})$ is a $100(1-\alpha)\%$-confidence interval for μ.*

Proof. We are required to prove

$$P(\mu \in (\overline{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\ ,\ \overline{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}})) = 1 - \alpha.$$

We have

$$\begin{aligned}
LHS &= P(\overline{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} < \mu, \mu < \overline{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}) \\
&= P(\overline{X} - \mu < t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\ ,\ -t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} < \overline{X} - \mu) \\
&= P(\overline{X} - \mu < t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\ ,\ \overline{X} - \mu > -t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}) \\
&= P((\overline{X} - \mu)/\frac{S}{\sqrt{n}} < t_{\alpha/2,n-1}\ ,\ (\overline{X} - \mu)/\frac{S}{\sqrt{n}} > -t_{\alpha/2,n-1}) \\
&= P(T < t_{\alpha/2,n-1}\ ,\ T > -t_{\alpha/2,n-1}) = P(-t_{\alpha/2,n-1} < T < t_{\alpha/2,n-1}) = 1 - \alpha
\end{aligned}$$

To prove the last equality draw a picture. □

Once we have an actual sample $x_1, x_2, \ldots, x_n$ we obtain the observed value $\overline{x}$ for the random variable $\overline{X}$ and the observed value s for the random variable S. We obtain the observed value (an ordinary interval) $(\overline{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}, \overline{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}})$ for the confidence (random) interval $(\overline{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\ ,\ \overline{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}})$ The observed value of the confidence (random) interval is also called the two-sided $100(1-\alpha)\%$ confidence interval for μ.

3 The lower-tailed confidence interval

In this section we will give the formula for the lower-tailed confidence interval for μ.

Theorem 3. *The random interval $(-\infty\ ,\ \overline{X} + t_{\alpha,n-1}\frac{S}{\sqrt{n}})$ is a $100(1-\alpha)\%$-confidence interval for μ.*

Proof. We are required to prove

$$P(\mu \in (-\infty\ ,\ \overline{X} + t_{\alpha,n-1}\frac{S}{\sqrt{n}})) = 1 - \alpha.$$

We have

$$\begin{aligned}
LHS &= P(\mu < \overline{X} + t_{\alpha,n-1}\frac{S}{\sqrt{n}}) = P(-t_{\alpha,n-1}\frac{S}{\sqrt{n}} < \overline{X} - \mu) \\
&= P(-t_{\alpha,n-1} < (\overline{X} - \mu)/\frac{S}{\sqrt{n}}) \\
&= P(-t_{\alpha,n-1} < T) \\
&= 1 - \alpha
\end{aligned}$$

To prove the last equality draw a picture - I want *you* to draw the picture on tests and the homework. □

Once we have an actual sample $x_1, x_2, \ldots, x_n$ we obtain the observed value $\overline{x}$ for the random variable $\overline{X}$ the observed value s for the random variable S and the observed value $(-\infty, \overline{x} + t_{\alpha,n-1}\frac{s}{\sqrt{n}})$ for the confidence (random) interval $(-\infty, \overline{X} + t_{\alpha,n-1}\frac{S}{\sqrt{n}})$. The observed value of the confidence (random) interval is also called the lower-tailed $100(1-\alpha)\%$ confidence interval for μ.

The random variable $\overline{X} + t_{\alpha,n-1}\frac{S}{\sqrt{n}}$ or its observed value the number $\overline{x} + t_{\alpha,n-1}\frac{s}{\sqrt{n}}$ is often called a confidence *upper bound* for μ because

$$P(\mu < \overline{X} + t_{\alpha,n-1}\frac{S}{\sqrt{n}}) = 1 - \alpha.$$

4 The upper-tailed confidence interval for μ

Problem Prove the following theorem.

Theorem 4. *The random interval* $(\overline{X} - t_{\alpha,n-1}\frac{S}{\sqrt{n}}\ ,\ \infty)$ *is a* $100(1-\alpha)\%$ *confidence interval for* μ.

The random variable $\overline{X} - t_{\alpha,n-1}\frac{S}{\sqrt{n}}$ or its observed value the number $\overline{x} - t_{\alpha,n-1}\frac{s}{\sqrt{n}}$ is often called a confidence *lower bound* for μ because

$$P(\mu > \overline{X} - t_{\alpha,n-1}\frac{S}{\sqrt{n}}) = 1 - \alpha.$$