STAT400 HOMEWORK ASSIGNMENT NUMBER 9

Pairs of discrete random variables

The point of this assignment is to understand the joint probability mass function and the correlation as measuring the *relationship* between two random variables X and Y defined on the same sample space.

Let X and Y be Bernoulli random variables with $p = 1/2$ defined on the same sample space. Hence we have

x	0	1
P(X=x)	1/2	1/2

and

y	0	1
P(Y=y)	1/2	1/2

However there are *infinitely many* possible ways in which X and Y can be related. These different ways are measured by the joint probability mass function $p_{X,Y}(x, y)$.

$x\backslash y$	0	1
0	a	b
1	c	d

So a, b, c, d are all between 0 and 1 and satisfy $a + b + c + d = 1$.

The Homework Problems

1. Show that d determines a, b and c. (Hint: since $X \sim Bin(1, 1/2)$ we have $a + b = 1/2$ and $c + d = 1/2$ and since $Y \sim Bin(1, 1/2)$ we have $a + c = 1/2$ and $b + d = 1/2$ - why is this?)

2. Find the covariance $Cov(X, Y)$ and the correlation $\rho_{X,Y}$ in terms of d.

3. Show that $Cov(X, Y) = 0$ implies that X and Y are independent. (this is highly exceptional - we will find an example in which $Cov(X, Y) = 0$ but X and Y are *not independent* in Problem 6.

4. Compute the covariance and correlation between X and Y for the following three joint probability mass functions.

A=

$x\backslash y$	0	1
0	1/2	0
1	0	1/2

B=

$x\backslash y$	0	1
0	1/4	1/4
1	1/4	1/4

C=

$x\backslash y$	0	1
0	0	1/2
1	1/2	0

5.Match each of the above three joint probability mass functions with the one of the following relationships between X and Y:

$$D = (X \text{ and } Y \text{ are independent}) \text{ or } E = (X = Y) \text{ or } F = (X = 1 - Y).$$

6. (the most important problem) Suppose now that we continue to assume that $X \sim Bin(1, 1/2)$ but we now assume that $Y \sim Bin(2, 1/2)$. So we have a new table

$x\backslash y$	0	1	2	
0	a	b	c	1/2
1	d	e	f	1/2
	1/4	1/2	1/4	

Note that we have now added the "margins" that tell you the distributions of X and Y. Find values for a, b, c, d, e, f so that

$$Cov(X, Y) = 0 \text{ but X and Y are not independent.}$$

Note this is a hard problem because a, b, c, d, e, f are all between zero and one and must satisfy

$$\begin{aligned} a + b + c &= 1/2 \\ d + e + f &= 1/2 \\ a + d &= 1/4 \\ b + e &= 1/2 \\ c + f &= 1/4 \end{aligned}$$

Also X and Y are not supposed to be independent. (Hint:make three of the entries in the probability mass function equal to zero)