Stat 400 **FINAL EXAM** **May 18, 2010**

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1. Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} (n+1)x^n, \ 0 \le x \le 1, \\ 0, otherwise. \end{cases}$$

Your answers to the next six parts should all be in terms of n.

(a) Find $E(X)$.

(b) Find $E(X^2)$.

(c) Find $V(X)$.

(d) Find $F(x)$, the cumulative distribution function of X.

(e) Find the median of X.

(f) Find the 90-th percentile of X.

(30 points)

2. A bus carrying n passengers makes three stops, we assume that each passenger on board is equally likely to get off at any given stop, and that the passengers act independently of each other.

(a) What is the probability that k passengers get off at the first stop?

(b) Suppose that four people ride the bus every Monday through Saturday, but only two ride on Sunday. On a certain day (picked at random), we observe that no one gets off at the first stop. In the light of this information what is the probability that the day is Sunday? (Hint: use Bayes' Theorem).

Bayes' Theorem

$$P(A|B) = (P(B|A)P(A))/(P(B|A)P(A) + P(B|A')P(A')).$$

(10 points)

3. Let S be the sample space of the experiment "deal three cards from a deck of cards without replacement". In all the parts of what follows "order counts" (unlike usual card hands).

(a) How many elements are in S?

(b) Let A be the subset of S where the first card is a heart. How many elements are in A? Compute $P(A)$.

(c) Let B be the subset of S such that the third card is the ace of hearts. How many elements are in B ? Be careful, you know the third card , that limits the choices for the first two cards (it is up to you to figure out how). Compute $P(B)$.

Suppose now we deal all 52 cards without replacement. So our new sample space S is the set of all arrangments of the 52 cards in the deck.

(d) Let C be the subset of S such that the last card is a heart. How many elements are in C ? Compute $P(C)$.

(20 points)

4. Suppose X and Y are random variables defined on the same sample space with the following joint probability mass function. The goal of this problem is to arrange that $Cov(X,Y) = 0$ but X and Y are independent.

X \ Y	-1	0	1
0	a	b	c
1	d	e	f

Here we assume that a, b, c, d, e, f are real numbers between zero and one such that $a + b + c + d + e + f = 1$.
(a) Give an example of a, b, c, d, e and f so that

$$Cov(X,Y) = 0.$$

Hint: if you make a clever choice of two of the numbers a,b,c,d,e,f to be zero you can make E(XY) = 0. Then if you make two of the remaining four nonzero numbers equal (again you need to make a clever choice) you can get $E(Y) = 0$.

(b) Now you have a lot of examples of pairs X and Y so that $Cov(X, Y) = 0$ *but for some of them X and Y will be dependent.* To finish the job, give definite numerical values for the remaining four nonzero numbers a, b, c, d, e, f (two of them are now equal) so that X and Y will be independent.

(15 points; 10 for (a) and 5 for (b))

5. Let $x_1, x_2, \cdots, x_n$ be a sample from a Bernoulli distribution with parameter p (so each x_i is either 1 or 0). Let $\hat{p}$ be the sample proportion (i.e. the number of 1's divided by n or equivalently the number of observed successes divided by the number of observations) and $\hat{q} = 1 - \hat{p}$. In the text it is stated that the interval

$$(\hat{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}\hat{q}}{n}}, \hat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}\hat{q}}{n}})$$

is the observed value of an approximate $100(1-\alpha)\%$ confidence interval for the population proportion (success probability) p.

A study of 100 football helmets of a certain type found that 10 showed damage when subjected to an impact test. Let p denote that true proportion of football helmets of this type that would be damaged by the impact test.

Use the above formula to construct a 90% confidence interval for p.
(Hint: think of the 100 helmets as a sample of size 100 from the population of all football helmets of the given type.)

The critical values z_α for the standard normal distribution

α	.1	.05	.025	.01
z_α	1.282	1.645	1.960	2.326

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(15 points)

6. Let $X_1, X_2, \cdots, X_n$ be a random sample from an exponential distribution with parameter λ. In what follows you may assume the following theorem (don't try to prove it).

Theorem A The random variable $V = 2n\lambda\overline{X}$ has chi-squared distribution with $2n$ degrees of freedom.

(a) Using Theorem A find a constant c (depending on α) such that the random interval

$$(c/(2n\overline{X}), \infty)$$

is a $100(1-\alpha)\%$ confidence interval for λ.

(b) Prove that your answer is correct.

(10 points)