Lecture 19: More Than Two Random Variables

Definition

If $X_1, X_2, \ldots, X_n$ are discrete random variables defined on the same sample space then their joint pmf is the function

$$P_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$$

If $X_1, X_2, \ldots, X_n$ are continuous then their joint pdf is the function $f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$ such that

Definition (Cont.)

for any region A in $\mathbb{R}^n$

$$P((X_1, X_2, \ldots, X_n) \in A) = \underbrace{\int \ldots \int_A f_{X_1,X_2,\ldots,X_n}(x_1, \ldots, x_n) dx_1, \ldots, dx_n}_{n\text{-fold multiple integral}}$$

Definition

The discrete random variables $X_1, X_2, \ldots, X_n$ *are independent if*

$$P_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = P_{X_1}(x_1)P_{X_2}(x_2)\ldots P_{X_n}(x_n).$$

Equivalently

$$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1)\ldots P(X_n = x_n)$$

The continuous random variables $X_1, X_2, \ldots, X_n$ are independent if

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_n}(x_n)$$

Definition

$X_1, X_2, \ldots, X_n$ are pairwise independent if each pair $X_i, X_j (i \neq j)$ is independent.
We will now see

Pairwise independence $\not\Longrightarrow$ *Independence*
of random variables $\Longleftarrow$ *of random variables*

First we will prove $\Longleftarrow$

Theorem

$X_1, X_2, \ldots, X_n$ independent $\Rightarrow X_1, X_2, \ldots, X_n$ are pairwise independent.

From now on we will restrict to the case $n = 3$ so we have THREE random variables X, Y, Z.

How do we get

$$P_{X,Y}(x,y) \quad \text{from} \quad P_{X,Y,Z}(x,y,z)$$

Answer (left to you to prove)

$$P_{X,Y}(x,y) = \sum_{\text{all } z} P_{X,Y,Z}(x,y,z) \qquad (\#)$$

Now we can prove X, Y, Z independent.

$$\implies \quad X, Y \text{ independent}$$

Since X, Y, Z are independent we have

$$P_{X,Y,Z}(x,y,z) = P_X(x)P_Y(y)P_Z(z) \qquad (\#\#)$$

Now play the RHS of $(\#\#)$ into the RHS of $(\#)$

$$\begin{aligned} P_{X,Y}(x,y) &= \sum_{\text{all } z} P_X(x)P_Y(y)P_Z(z) \\ &= P_X(x)P_Y(y) \underbrace{\sum_{\text{all } z} P_Z(z)}_{=1} \\ &= P_X(x)P_Y(y) \end{aligned}$$

This proves X and Y are independent Identical proofs prove the pairs X, Z and Y, Z are independent.

Now we construct X, Y, Z (actually X_A, X_B, X_C) so that each *pair* is independent but the triple X, Y, Z is *not independent*.

The multinomial coefficient

The multinomial coefficient $\binom{n}{k_1, k_2, \cdots, k_r}$ is defined by

$$\binom{n}{k_1, k_2, \cdots, k_r} = \frac{n!}{k_1! k_2! \cdots k_r!}$$

Suppose an experiment has r outcomes denoted $1, 2, 3, \cdots, r$ with probabilities $p_1, p_2, \cdots, p_r$ respectively. Repeat the experiment n times and assume the trials are independent.

$\binom{n}{k_1, k_2, \cdots, k_r}$

A Variation on the Cool Counter example

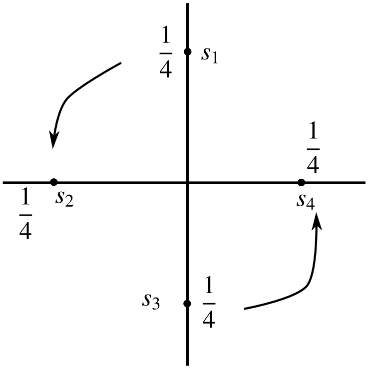
Lets go back to the “cool counter example”, Lecture 16, page 18 of three events A, B, C which are pairwise independent but no independent so

$$P(A \cap B \cap C) \neq P(A)P(B)P(C)$$

The idea is to convert the three events to random variables X_A, X_B, X_C so that $X_A = 1$ on A and O on A’ etc.

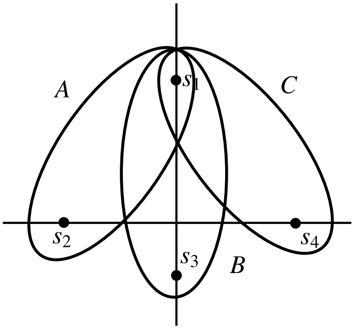
In fact we won't need the corner points $(-1,-1)$, $(-1,1)$, $(1,-1)$ and $(1,1)$ we put $S_1 = (0,1)$, $S_2 = (-1,0)$, $S_3 = (0,1)$, $S_4 = (1,0)$ and retain their probabilities so

$$P(\{S_j\}) = \frac{1}{4}, \quad 1 \leq j \leq 4$$



We define

$$A = \{s_1, s_2\}$$
$$B = \{s_1, s_3\}$$
$$C = \{s_1, s_4\}$$



We define X_A, X_B, X_C on S by

$$X_A(s_j) = \begin{cases} 1, & \text{if } s_j \in A \\ 0, & \text{if } s_j \notin A \end{cases}$$

$$X_B(s_j) = \begin{cases} 1, & \text{if } s_j \in B \\ 0, & \text{if } s_j \notin B \end{cases}$$

$$X_C(s_j) = \begin{cases} 1, & \text{if } s_j \in C \\ 0, & \text{if } s_j \notin C \end{cases}$$

So $$P(X_A = 1) = P(\{S_1, S_2\}) = \frac{1}{2}$$

$$P(X_A = 0) = P(\{S_3, S_4\}) = \frac{1}{2}$$

and similarly for X_B and X_C.

So X_A, X_B and X_C

are Bernoulli random variables

Let's compute the joint pmf of X_A and X_B. We know the margin

X_A \ X_B	0	1	
0			$1/2$
1			$1/2$
	$1/2$	$1/2$	

The subset where $X_A = 1$ is the subset $\{s_1, s_2\}$ so we write an equality of events

$$(X_A = 1) = \{s_1, s_2\}$$

Similarly

$$(X_A = 0) = \{s_3, s_4\}$$
$$(X_B = 1) = \{s_1, x_3\}, (X_B = 0) = \{s_2, s_4\}$$
$$(X_C = 1) = \{s_1, s_4\}, (X_C = 0) = \{s_2, s_3\}$$

Hence

$$(X_A = 0) \cap (X_B = 0) = \{S_4\}$$

$$\text{so} \quad P(X_A = 0, X_B = 0) = \frac{1}{4}$$

$$(X_A = 0) \cap (X_B = 1) = \{S_3\}$$

$$\text{so} \quad P(X_A = 0, X_B = 1) = \frac{1}{4}$$

$$(X_A = 1) \cap (X_B = 0) = \{S_2\}$$

$$P(X_A = 1, X_B = 0) = \frac{1}{4}$$

$$(X_A = 1) \cap (X_B = 1) = \{S_1\}$$

$$P(X_A = 1, X_B = 1) = P(\{S_1\}) = \frac{1}{4}$$

etc.

So the joint proof of X_A and X_B is

X_A \ X_B	0	1	
0	1/4	1/4	1/2
1	1/4	1/4	1/2
	1/2	1/2	

so X_A and X_B are independent. The same is true for X_A and X_C and X_B and χ_C. Now we show the triple X_A, X_B and X_C is NOT independent.

We will show

$$P(X_A = 1, X_B = 1, X_C = 1)$$
$$\neq P(X_A = 1)P(X_B = 1)P(X_C = 1)$$

The RHS $= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$

The left-hand side is the probability of the event

$$\begin{aligned}&(X_A = 1) \cap (X_B = 1) \cap (X_C = 1)\\ &= \{S_1, S_2\} \cap \{S_1, S_3\} \cap \{S_1, S_4\}\\ &= \{S_1\}.\end{aligned}$$

So

$$P(X_A = 1, X_B = 1, X_C = 1) = P(\{S_1\}) = \frac{1}{4}$$

so

$$\text{LHS} = \frac{1}{4}$$
$$\text{RHS} = \frac{1}{8}$$

Remark

This counter example is more or less the some as the "cool counter example". We just replaced (more or less) A, B, C by their "characteristic functions".