Lecture 2 : Counting Techniques

2.3 The Three Basic Rules

1. The Product Rule for Ordered Pairs and Ordered k-tuples

Our first counting rule applies to any situation in which a set consists of *ordered* pairs of objects (a, b) where a comes from a set B.

In terms of pure mathematics the *Cartesian product* $A \times B$ is the set of such pairs

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Proposition (text pg. 60)

If the first element of the ordered pair can be selected in n_1 ways and if for each of these n_1 ways the second element can be selected in n_2 ways then the number of pairs is $n_1 n_2$.

Mathematically - If $\sharp(A) = n_1$ and $\sharp(B) = n_2$ then $\sharp(A \times B) = n_1 n_2$. There are analogous results for ordered triples etc.

$$\sharp(A \times B \times C) = n_1 n_2 n_3$$

Example

How many "words" of two letters can we make from the alphabet of five letters $\{a, b, c, d, e\}$.

Solution

Note that order counts $ab \neq ba$.
There are two ways to think about the problem pictorially.

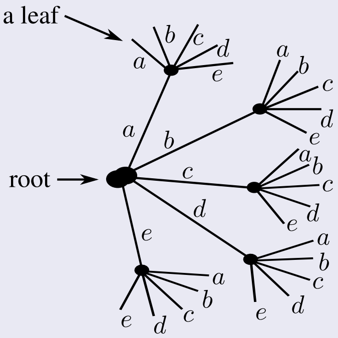
1. Filling in two slots _ _

We have a choice of 5 ways to fill in the first slot and for each of these we have 5 more ways to fill in the second slot so we have 25 ways.

$$\underline{5}\ \underline{5} = 25$$

Solution (Cont.)

2. Draw a tree where each edge is a choice



The number of pairs is the number paths from the root to a "leaf" (i.e., a node at the far right).
In this case there are 25 paths.

Problem

How many words of length 3?

2. Permutations (pg. 62)

In the previous problem the word *aa* was allowed. What if we required the letters in the word to be distinct. Then we would get 2-permutations from the 5-element set $\{a, b, c, d\}$ according to the following definition.

Definition

An ordered sequence of k distinct objects taken from a set of n elements is called a k-permutation of the n objects. The number of k-permutations of the n objects will be denoted $P_{k,n}$.

So *order counts*

Let us return to our 5 element set $\{a, b, c, d, e\}$ and count the number of 2-permutations.

It is best to think in terms of slots

$$\underline{5}\ \underline{4} = 20$$

There are 5 choices for the first slot but only 4 for the second because whatever we put in the first slot cannot be put in the second slot so $P_{2,5} = 20$.

What is $P_{3,5}$?

Proposition (pg. 68)

$$P_{k,n} = \underbrace{n(n-1)(n-2)\dots(n-k+1)}_{k \ \ terms}$$

Proof.

Fill in k slots with no repetitions

how many

how many

$$\underbrace{\underline{n}\ \underline{n-1}\ \_\ \_\ \_}_{k}$$

Note that if we allowed repetitions we would get n^k

$$\underbrace{\underline{n}\ \underline{n}\ \underline{n}\ \_\ \underline{n}}_{k}$$

□

There is a very important spacial case

$$P_{n,n} = n! = n(n-1)(n-2)\dots 3.2.1$$

There are $n!$ ways to take n distinct objects and arrange them in order.

Example

$n = 3$, $\{a, b, c\}$

$$\left.\begin{matrix} abc \\ acb \\ bac \\ bca \\ cab \\ cba \end{matrix}\right\} \quad 3! = (3)(2)(1) = 6$$

When you list objects it is helpful to list them in dictionary order.

A Better Formula for $P_{k,n}$

Here is a better formula for $P_{k,n}$.

Proposition

$$P_{k,n} = \frac{n!}{(n-k)!}$$

Proof.

This is an algebraic trick

$$\frac{n!}{(n-k)!} = \frac{n(n-1)\dots(n-k+1)\overbrace{(n-k)(n-k-1)\dots 3.2.1}^{(n-k)!}}{(n-k)!}$$

So cancel the second part of the numerator with the denominator

$$\frac{n!}{n-k!} = n(n-1)\dots(n-k+1) = P_{k,n}$$

$\square$

The Birthday Problem

Suppose there are n people in a room. What is the probability B_n that at least two people have the same birthday (eg., March 11)? Let S be the set of all possible birthdays for then n people so

$$\sharp(S) = (365)^n$$

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{n \text{ people}}$$

(we ignore leap-years so this isn't quite right)

Now let $A \subset S$ be the event that at least two people have the same birthday. So A' = all the people in the room have different birthdays.
So

$$B_n = 1 - P(A')$$

Now what is $P(A')$? Order the people. The total number of people with different birthdays is $P_{n,365}$ = the number of n-permutations of a set with 365 elements. So

$$\sharp(A') = P_{n,365} = \frac{n!}{n-k!} = n(n-1)\dots(n-k+1)$$

.

So

$$B_n = 1 - \frac{P_{n,365}}{(365)^n}$$

Combinations

There are many counting problems in which one is given a set of n objects and one wants to count the number of *unordered* subsets with k elements.

An unordered subset with k elements taken from a set of n elements is called a k-combination of that set. The number of k-combinations is denoted $C_{k,n}$.

Which is bigger $C_{k,n}$ or $P_{k,n}$?

What is $C_{n,n}$?

Example

$P_{2,3} = 6$, $C_{2,3} = 3$
$S = \{a, b, c\}$

2 permutations of S	*2 combinations of S*
$ab \quad ba$	$\{a, b\}$
$bc \quad cb$	$\{b, c\}$
$ac \quad ca$	$\{a, c\}$

Each two combination gives rise to 2. 2-permutations.
So

$$P_{2,3} = 2C_{2,3} = (2)(3) = 6$$

A Formula for $C_{k,n}$

Proposition (pg. 64)

$$P_{k,n} = C_{k,n} \cdot k! \quad So$$

$$C_{k,n} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

Proof.

To make a k-permutation first make an unordered choice of the k-elements i.e., choose a k-combination, then, for each such choice arrange the elements in order (there are $P_{k,k} = k!$ ways to do this). So we have

$$\sharp(k\text{-permutations}) = \sharp(k\text{-combinations}).k!$$

$\square$

More notation

The binomial coefficient $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

This is because a famous result from the past is

$$\underbrace{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}_{\text{The binomial theorem.}}$$

But a counting argument gives that coefficient of $a^k b^{n-k}$ is also equal to $C_{k,n}$. I will leave that argument to you with the hint that you should write $(a+b)^n$ as $\underbrace{(a+b)\dots(a+b)}_{n \text{ terms}}$ and make a choice of either a or b from each bracket.

We will use the symbol $\binom{n}{k}$ in place of of symbol $C_{k,n}$.

More Problems

1. How many 5 card poker hands are there?
2. How many 13 card bridge hands are there?

Lastly

Proposition

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof.

Challenge.
Find two proofs, one "combinatorial" and one algebraic. □