Lecture 3

Probability Computations

Bridge Hands and Poker Hands

Bridge Hands

If you play bridge you get dealt a hand of 13 cards.
Let S be the set of all bridge hands (so our "experiment" is dealing 13 cards).
Since the order in which you receive the cards doesn't count (it never does in card games)

$$\sharp(S) = \binom{52}{13}$$
$$= \text{the number of 13 element subsets of a 52 element set.}$$

We will now compute the probability of certain bridge hands.

1. Let A = the hand is all hearts. What is $\sharp(A)$? We use the "principle of restricted choice". Our choice is restricted to the subset of hearts - we here to choose 13. There are 13 hearts so we have $\binom{13}{13} = 1$ hands.
   So
   $$P(A) = \frac{\sharp(A)}{\sharp(S)} = \frac{1}{\binom{52}{13}}$$
   (A is very unlikely)

2. Let B = there are no hearts. We want to compute $P(B)$. There is a point here you need to be careful about. It is not true that B is the complement of A. The complement of B is that there is at least one heart.
What is $\sharp(B)$?
Once again we use the "principle of restricted choice" (I made up this name, it isn't in common usage). We have to choose 13 non hearts. There are $52 - 13 = 39$ non hearts so we have to choose 13 things from 39 things so

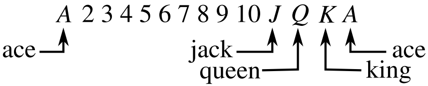
$$\sharp(B) = \binom{39}{13}$$

So

$$P(B) = \frac{\binom{39}{13}}{\binom{52}{13}}$$

Poker Hands

If you play poker you get dealt a hand of 5 cards (or sometimes 7 cards). For the next few examples we will assume we are playing "5-card poker". There is also the role of aces. In your first homework (" first poker problem "), #43 aces can be "either high or low".. This means that you can count them as 1's (i.e., lower than anything else) or higher than anything else so we have in order



There are $\binom{52}{5}$ (five-card) poker hands.

1 $A = \underbrace{A \text{ "Straight"}}_{\substack{\text{all five cards are} \\ \text{consecutive = a "straight"}}}$

e.g., 56789.

or $A2345$ aces are low

or $10JQKA$ aces are high.

Find $P(A)$, this is Problem 43 from the text.

There is one observation you need -

a straight is almost determined by its bottom (i.e., lowest) card.

You need one more sub observation: the cards, $\underline{J, Q, K}$ cannot be bottom cards for a straight (because there aren't enough cards above them).
The last straight is
$10 \;\; J \;\; K \;\; Q \;\; A \longleftarrow$ aces are high.
Also A can be a bottom card because aces are low

$$A \;\; 2 \;\; 3 \;\; 4 \;\; 5$$

So there are $13 - 3 = 10$ bottom cards so there are 10 different kinds of straights - so at first glance one might think there are 10 straights. But, let's consider $A2345$. We haven't taken account of the SUITS of the cards. The A could be any one of four suits, for each of these the 2 could be any one of four suits so there are 4^5 straights of the form $AZ345$. Hence

$$\sharp(A) = (\underbrace{10}_{\text{lowest card}})(\underbrace{4^5}_{\text{suit of each card}})$$

So

$$P(A) = \frac{(10)(4^5)}{\binom{52}{5}}$$

2. A "Flush"

A "flush" is a poker hand in which all cards are of the same suit (spade, heart, diamond or club).

Let B = set of all flushes.

First - how many hands are there that are all hearts?

We use the principle of restricted choice. There are 13 hearts, we have to choose 5. So there are $\binom{13}{5}$ such hands. So we have

$$\sharp(B) = \underbrace{4}_{\text{choose a suit}} \underbrace{\binom{13}{5}}_{\text{pick 5 cards from it}} = (4)\binom{13}{5}$$

So

$$P(B) = \frac{(4)\binom{15}{5}}{\binom{52}{5}}$$

3. A "Straight Flush"

Lets combine 2 and 3. So

C = set of all straights so that all the cards have the same suit.

To compute $\sharp(C)$ first pick the lowest card 10 ways

then

pick the suit that all five cards have 4 ways

So $\sharp(C) = (10)(4) = 40$

$$\text{and} \quad P(C) = \frac{40}{\binom{52}{5}} = \text{a very small number}$$

In the next two problems we will see there is often a tricky decision to make: do we take ordered pairs (triples . . .) or unordered pairs (triples . . .)?

4. A Full House

A full house is a poker hand which consists of 3 of one kind and 2 of a different kind e.g., *JJJ KK*
Let D = set of full houses.
Here is how we compute $\sharp(D)$.

1. Pick an **ordered** pair of kinds e.g., J or K.

2. Pick 3 of the first kind and 2 of the second kind.

So in the above case we get $JJJ\ KK$.

Key test to perform

Were we right when we said ORDERED pair above?
So let's test, reverse the order K, J and check if we get a different hand.
If we do we were right to say ORDERED pairs.
If we don't we were wrong and we have to replace ORDERED pairs in 1. by UNORDERED pairs.

Okay if we take the pair K, J and do 2 we get the hand $KKK\ JJ$. Are $KKK\ JJ$ and $JJJ\ KK$ different poker hands ? *Yes*, the first beats the second so we were right to pick ordered pairs.
Now lets finish the job

1. There are 13 kinds, so ordered pair of kinds is a 2 permutation of the 13 element set of kinds.
   So.

$$\sharp(1.) = P_{2,13} = \frac{(13)!}{(11)!}$$

$$= (13)(12)$$

2. Now there are $\binom{4}{3}$ to pick the first kind (say the first kind is a jack so we have to pick 3 of the 4 jacks) and $\binom{4}{2}$ ways to pick the second kind so

$$\sharp(D) = \underbrace{(13)(12)}_{\text{2 kinds}} \binom{4}{3}\binom{4}{2}$$

so

$$P(D) = \frac{(13)(12)\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

4. Two Pair

In poker "two pair" is a hand which consists of 2 of one kind, 2 of a second (*different*) kind and one more of a third kind (*different* from the other two)
e.g., $JJ\ KK10$
Let E be the set of two - pair hands. Here is how we compute $\sharp(E)$.

1. Pick an $\begin{cases} \textit{ordered} \\ \textit{unordered} \end{cases}$ pair of kinds.
   e.g., J, K.
2. Pick a third kind.
3. Pick two cards from each of the first two kinds and one of the

third kind.

64,000 Question

In 1. do we pick an *ordered* pair or an *unordered* pain? ¡br¿
Do the test
The pair $J, K \longrightarrow JJKK$
The pair $K, J \longrightarrow KkJJ$
In poker *JJKK* and *KKJJ* are the *SAME* so J, K and K, J give the *SAME* hand so order does not matter.
So we pick an *unordered* pair from which to choose the first four cards.

Now we compute $\sharp(E)$.

1. There are 13 kinds.
   We want the number of **unordered** pairs. It is
   $\binom{13}{2} = \frac{(13)(12)}{2} = 78$
   This number is one half times the number we got for Step 1. in the full house case.
2. We have chosen two kinds. There are $13 - 2 = 11$ left. We have to pick one so $\binom{11}{1} = 11$.

3. $\binom{4}{2} = 6$ for the first kind.
   $\binom{4}{2} = 6$ for the second kind.

(there is no first or second but it does matter the numbers are the same) $\binom{4}{1} = 4$ for the third kind.

$$\sharp(t) = \left(\frac{(13)(12)}{2}\right)(11)\binom{4}{2}\binom{4}{2}(4)$$

so

$$P(E) = \frac{\left(\frac{(13)(12)}{2}\right)(11)(6)(6)(4)}{\binom{52}{5}}$$