Lecture 7
The Five Basic Discrete Random Variables

In this lecture we define and study the five basic discrete random variables.

The Five Basic Discrete Random Variables

1. Binomial
2. Hypergeometric
3. Geometric
4. Negative Binomial
5. Poisson

Remark

On the handout "The basic probability distributions" there are six distributions. I did not list the Bernoulli distribution above because it is too simple.

In this lecture we will do 1. and 2. above.

The Binomial Distribution

Suppose we have a Bernoulli experiment with $P(S) = P$, for example, a weighted coin with $P(H) = p$. As usual we put $q = 1 - p$.

Repeat the experiment (flip the coin). Let $X = \sharp$ of successes ($\sharp$ of heads).

We want to compute the probability distribution of X. Note, we did the special case $n = 3$ in Lecture 6, pages 4 and 5.

Clearly the set of possible values for X is $0, 1, 2, 3, \ldots, n$.
Also

$$P(X = 0) = P(TT \quad T) = qq \ldots q = q^n$$

Explanation

Here we assume the outcomes of each of the repeated experiments are *independent* so

$$P((T \text{ on } 1^{\text{st}}) \cap (T \text{ on } 2^{\text{nd}}) \cap \cdots \cap (T \text{ on} n\text{-th})$$
$$P(T \text{ on } 1^{\text{st}})P(T \text{ on } 2^{\text{rd}}) \ldots P(T \text{ on } n\text{-th})$$
$$q\, q \ldots q = q^n$$

Note T on 2^{nd} means T on 2^{nd} *with no other information* so

$$P(T \text{ on } 2^{\text{nd}}) = q.$$

Also

$$P(X = n) = P(HH \dots H) = p^n$$

Now we have to work
What is $P(X = 1)$?

Another standard mistake

The events $(X = 1)$ and $\underbrace{HTT \dots T}_{n-1}$ are NOT equal.

Why - the head doesn't have to come on the first toss

So in fact

$$(X = 1) = (HTT \dots T) \cup (THT \dots T) \cup \cdots \cup (TTT \dots TH)$$

All of the n events on the right have the same probability namely pq^{n-1} and they are mutually exclusive. There are n of them so

$$P(X = 1) = npq^{n-1}$$

Similarly

$$P(X = n - 1) = npq^{n-1}$$

(exchange H and T above)

The general formula

Now we want $P(X = k)$
First we note

$$P(\underbrace{H \dots H}_{k} \ \underbrace{TT \dots T}_{n-k}) = p^k q^{n-k}$$

But again the heads don't have to come first. So we need to

(1) Count all the words of length n in H and T that involve k H's and $n - k$ T's.

(2) Multiply the number in (1) by $p^k q^{n-k}$.

So how do we solve 1. Think of filling n slot's with k H's and $n - k$ T's

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}$$

Main Point

Once you decide where the k H's go you have no choice with the T's. They have to go in the remaining $n - k$ slots.
So choose the k-slots when the heads go. So we have to make a choose of k things from n things so $\binom{n}{k}$.

So,

$$P(X = k) = \binom{n}{k} p^k q^{n-k}$$

So we have motivated the following definition.

Definition

A discrete random variable X is said to have binomial distribution with parameters n and p (abbreviated $X \sim \mathrm{Bin}(n, p)$)
If X takes values $0, 1, 2, \ldots, n$ and

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, 0 \leq k \leq n. \qquad (*)$$

Remark

The text uses x instead of k for the independent (i.e., input) variable. So in the text this would be written

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

I like to save x for the variable case of continuous random variables however I will sometimes use x in the discrete case too.

Finally we may write

$$p(k) = \binom{n}{k} p^k q^{n-k}, \quad 0 \le k \le n \qquad (**)$$

The text uses $b(\cdot, n, p)$ for $p(\cdot)$ so would write for (**)

$$b(k, n, p) = \binom{n}{k} p^k q^{n-k}$$

The Expected Value and Variance of a Binomial Random Variable

Proposition

Suppose $X \sim \mathrm{Bin}(n, p)$. Then $E(X) = np$ and $V(X) = npq$ so σ = standard deviation $= \sqrt{npq}$.

Remark

The formula for $E(X)$ is what you might expect. If you toss a fair coin 100 times the $E(X) =$ expected number of heads $np = (100)\left(\frac{1}{2}\right) = 50$.

However if you toss it 51 times then $E(X) = \frac{51}{2}$ - not what you "expect".

Using the binomial tables

Table A1 in the text
pg. A2,A3,A4 tabulate the cdf $B(x, n, p) = P(X \leq x)$ for $n = 5, 10, 15, 20, 25$ and selected values of p.

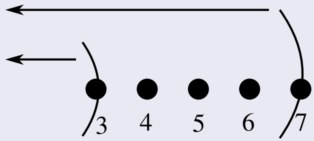
Example (3.32)

Suppose that 20% of all copies of a particular text book fail a certain binding strength text. Let X denote the number among 15 randomly selected copies that fail the test. Find

$$P(4 \leq X \leq 7).$$

Solution

$X \sim \mathrm{Bin}(15, .2)$. We want to compute $P(4 \leq X \leq 7)$ using the table on page 664. So how to we write $P(4 \leq X \leq 7)$ in terms of terms of the form $P(X \leq a)$



In the figure $P(X \leq 3)$ is the region to the left of the left-most arc and $P(X \leq 7)$ is the region to the left of the right-most arc.

Answer

$$(\sharp) P(4 \leq X \leq 7) = P(X \leq 7) - P(X \leq 3)$$

So

$$P(4 \leq X \leq 7) = B(7, .15, .2) - B(3, .15, .2)$$

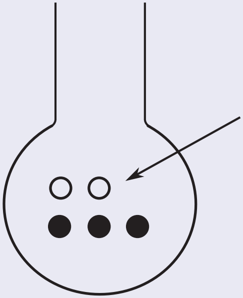
from table

$$= .996 - .648$$

N.B. *Understand $(\sharp)$. This the key using computers and statistical calculators to compute.*

The hypergeometric distribution

Example



Consider an urn containing N chips of which M are black and $L = N - M$ are white. Suppose we remove n chips without replacement so $n \leq N$.
In the figure there are 3 black chips and 2 white chips so in the picture $N = 5, M = 3$ and $L = 2$.
Define a random variable X by $X = \sharp$ of black chips we get.

Find the probability distribution of X.

Proposition

$$P(X = k) = \frac{\binom{M}{k}\binom{L}{n-k}}{\binom{N}{n}} \qquad (*)$$

if

$$(b) \quad \underbrace{\max(0, n - L) \leq k \neq \min(n, M)}$$

This means k $\leq$ both n and M and both 0 *and* $n - L \leq k$.
These are the possible values of k, *that is, if* k *doesn't satisfy (b) then*

$$P(X = k) = 0.$$

Proof of the formula (*)

Suppose we first consider the special case where all the chips are black so

$$P(X = n).$$

This is the same problem as the one of finding all hearts in bridge.

$$\text{black chip} \longleftrightarrow \text{heart}$$
$$\text{white chip} \longleftrightarrow \text{non heart}$$

So we use the principle of restricted choise

$$P(X = n) = \frac{\binom{M}{n}}{\binom{N}{n}}$$

This agrees with (*).

But (*) is harder because we have to consider the case where there are $k < n$ black chips. *So we have to choose $n-k$ white chips as well.*
So choose k black chips, there are $\binom{M}{k}$ ways, then for each such choice, choose $n-k$ white chips, there are $\binom{L}{n-k}$ ways.
So

$$\sharp \left\{ \begin{array}{l} \text{choices of } \textit{exactly} \\ k \text{ black chips} \\ \text{in the } n \text{ chips} \end{array} \right\} = \binom{M}{k}\binom{L}{n-k}$$

Clearly there are $\binom{N}{n}$ ways of choosing n chips from N chips so (*) follows.

Definition

If X is a discrete random variable with pmf defined by the formula in the previous Proposition then X is said to have hyper geometric distribution with parameters n, M, N. In the text the pmf is denoted

$$h(x; n, M, N).$$

What about the conditions

$$\max(0, n - L) \leq k \leq \min(n, M) \tag{b}$$

This really means

$$k \leq \text{ both } n \text{ and } M \tag{b_1}$$

and

$$k \geq \text{both } 0 \text{ and } n - L \tag{b_2}$$

(b_1) says

$k \leq n \quad \longleftrightarrow$ we can't choose more then n black chips because we are only choosing n chips in total

$k \leq M \quad \longleftrightarrow$ because there are only M black chips to choose from

(b_2)

$$k \geq 0 \text{ is obvious and } k \geq n - L \text{ follows because } k = n - L$$

So the above three inequalities are necessary. At first glance they look sufficient because if k satisfies the above three inequalities you can certainly go ahead and choose k black chips.

But what about the white chips? We aren't done yet, you have to choose $n - k$ white chips and there are only L white chips available so if $n - k > L$ we are sun k.

So we must have

$$n - k \leq L \Leftrightarrow k \geq n - L$$

This is the second inequality of (b_2). If it is satisfied we can go ahead and choose the $n - k$ white chips so the inequalities in (b) are necessary and sufficient.

Proposition

Suppose X has hypergeometric distribution with parameters n, M, N. Then

(i) $E(X) = n\dfrac{M}{N}$

(ii) $V(X) = \left(\dfrac{N-n}{N-1}\right) n\dfrac{M}{N}\left(1 - \dfrac{M}{N}\right)$

If you put

$$p = \frac{M}{N} = \begin{array}{l}\textit{the probability of getting}\\ \textit{a black chip on the first draw}\end{array}$$

then we may rewrite the above formulas as

$$\left.\begin{aligned} E(X) &= np \\ V(X) &= \left(\frac{N-n}{N-1}\right) npq \end{aligned}\right\} \begin{array}{l}\textit{reminiscent}\\ \textit{of the}\\ \textit{binomial}\\ \textit{distribution}\end{array}$$

Another way to Derive (*)

There is another way to derive (*) - the way we derived the binomial distribution. It is way harder.

Example

Take $n = 2$

$$P(X=0) = \frac{L}{N}\frac{L-1}{N+1}$$

$$P(X=2) = \frac{M}{N}\frac{M-1}{N-1}$$

$$P(X=1) = P(RW) + P(WR)$$

$$= \frac{M}{N}\frac{L}{N-1} + \frac{L}{N}\frac{M}{N-1}$$

$$= 2\frac{M}{N}\frac{L}{N-1}$$

$$= 2\frac{M}{N}\frac{L}{N-1}$$

In general, we claim that all the words with k B's and $n-k$ W's have the some probability. Indeed each of these probabilities are fractions with the same denominator

$$N(N-1)\dots(N-n-1)$$

and they have the same factors in the numerator scrambled up

$$M(M-1)(M-L+1) \quad \text{and} \quad L(L-1),\dots,(L-n-k+i)$$

But the order of the factors doesn't matter so

$$\begin{aligned} P(X=k) &= \binom{n}{k} P(\overbrace{R\dots R}^{k} W\dots W) \\ &= \binom{n}{k} \frac{M(M-1)\dots(M-k+1)L(L-1)\dots(L-n-k+1)}{N(N-1)\dots N(-n+1)} \end{aligned}$$

Why is (*) equal to this?

$$(*) = \frac{\binom{M}{k}\binom{L}{n-k}}{\binom{N}{n}}$$

cancelling $(M-k)!$

cancelling $(L-n-k+1)$

$$= \frac{\frac{M(M-1)\ldots(M-k+1)}{k!}\;\frac{L(L-1)\ldots(L-n-k+1)}{(n-k)!}}{\frac{N(N-1)\ldots(N-n+1)}{n!}}$$

goes on top

$$(*) = \frac{\binom{M}{k}\binom{L}{n-k}}{\binom{N}{n}}$$

$$= \frac{\frac{M(M-1)\ldots(M-k+1)}{k!}\;\frac{L(L-1)\ldots(L-n-k+1)}{(n-k)!}}{\frac{N(N-1)\ldots(N-n+1)}{n!}}$$

exercise in fractions

$$= \frac{n!}{k!(n-k)!}\frac{M(M-1)\ldots(M-k+1)L(L-1)\ldots(L-n-k+1)}{N(N-1)\ldots(N-n+1)}$$

$$= \binom{n}{k}\frac{M(M-1)\ldots(M-k+1)L(L-1)\ldots(L-n-k+1)}{N(N-1)\ldots(N-n+1)}$$

Obviously, the first way (*) is easier so if you are doing a real-world problem and you start getting things that look like (**) step back and see if you can use the first method instead. You will tend to try the second method first. I will test you on this later.

Prediction (I was wrong before)
Most of you will use the second (wrong) method.

An Important General Problem

Suppose you draw n chips *with replacement* and let X be the number of black chips you get. What distribution does X have?

This explains (a little) the formulas on page 21. Note that if N is far bigger than n then it is almost like drawing with replacement. "The urn doesn't notice that any chaps have been removed because so few (relatively) have been removed."

In this case

$$\frac{N-n}{N-1} = \frac{N\left(1-\frac{n}{N}\right)}{N\left(1-\frac{1}{N}\right)} \approx \frac{N}{N} = 1$$

(because N is huge $\dfrac{1}{N}$ and $\dfrac{n}{N}$ are approximately 0)

So $V(X) \approx npq$

The number $\dfrac{N-n}{N-1}$ is called the "finite population correction factor".