HW1, due Wednesday, September 9
Math 600, Fall 2023
Patrick Brosnan, Instructor

Practice Problems and reading: Read Sections 1 and 2 of Chapter I of Aluffi's book. Then do the following problems from Aluffi for practice, but do not turn them in.

I.1) 4
I.2) 7

The format is that "1.1" means the exercises from Section 1 of Chapter I in Aluffi's book (which start on page 8).

Graded Problems: Work the following problems for a grade. Turn them in on Canvas.

1. (4 points each) Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \sin x$.

(1) Is f one-one?
(2) Is f onto?
(3) What is $f(\mathbb{R})$?
(4) What is $f([0, 10])$?
(5) What is $f^{-1}([-1, 1])$?
(6) What is $f^{-1}([2, \infty))$?

2. (4 points each) Which of the following are equivalence relations on $\mathbb{Z}$?

(1) The set $R = \{(x, y) \in \mathbb{Z}^2 : 2 \mid x^2 - y^2\}$. (Here $a \mid b$ means that a divides b.)
(2) The set $R = \{(x, y) \in \mathbb{Z}^2 : x - y \geq 0\}$.
(3) The set $R = \{(x, y) \in \mathbb{Z}^2 : \sin(x\pi/5) = \sin(y\pi/5)\}$.
(4) The set of all pairs $(x, y) \in \mathbb{Z}^2$ such that $x - y = z^3$ for some $z \in \mathbb{Z}$.

3. (20 points) Suppose $f : X \to Z$ and $g : Y \to Z$ are mappings. The *fiber product* of f and g is the set

$$X \times_Z Y := \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

One particular example of this is when $X = Y$ and $f = g$. Then I write $K(f) := X \times_Y X$. Some authors call $K(f)$ the *kernel pair* of f.

Show that, for any mapping $f : X \to Z$, $K(f)$ is an equivalence relation on X.

4. (20 points) Suppose X and Y are sets. I write $\mathrm{Fun}(X,Y)$ for the set of all functions from X to Y.

(1) Suppose $X = \{1,2,3\}$ and $Y = \{0,1\}$. Write down all elements of $\mathrm{Fun}(X,Y)$. If $f : X \to Y$ is a function, then you can write it explicitly by writing down the triple $(f(1), f(2), f(3))$.
(2) Suppose X has n elements and Y has m elements, with n and m both nonnegative integers. What is the cardinality of the set $\mathrm{Fun}(X,Y)$? In other words, how many elements does it have?

5. Suppose X,Y,Z and W are all sets. Suppose $g : Y \to Z$ is a function in $\mathrm{Fun}(Y,Z)$. We can define two new functions $g_* : \mathrm{Fun}(X,Y) \to \mathrm{Fun}(X,Z)$ and $g^* : \mathrm{Fun}(Z,W) \to \mathrm{Fun}(Y,W)$ as follows:

$$g_*(f) := g \circ f$$
$$g^*(h) = h \circ g.$$

The map g_* is sometimes called the *pushforward function* and the map g^* is sometimes called the *pullback function*.

(1) (10 points) If g is one-one, show that g_* is also one-one.
(2) (10 points) If g is onto, show that g^* is one-one.
(3) (10 point bonus) If g_* is one-one for every set X, show that g is one-one.
(4) (10 point bonus) If g^* is one-one for every set W, show that g is onto.