HW2, due Wednesday, September 16
Math 600, Fall 2023
Patrick Brosnan, Instructor

Practice Problems and Reading: Read Sections 3 and 4 of Chapter I of Aluffi's book. Then do the following problems from Aluffi for practice, but do not turn them in.

I.3) 1, 2, 8
I.4) 1, 2, 3

The format is that "1.1" means the exercises from Section 1 of Chapter I in Aluffi's book (which start on page 8).

Terminology: In problem 2 I use the word *fiber*. If $f : X \to Y$ is a map, then the *fiber* of f over an element $y \in Y$ is the preimage $f^{-1}(\{y\})$. For example, if $f : \mathbb{R}^2 \to \mathbb{R}$ is the map $(x, y) \mapsto x^2 + y^2$, then the fiber of f over 1 is the unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Graded Problems: Work the following problems for a grade. Turn them in on Canvas.

1. (**25 points**) If X is a set, write $\mathrm{Equiv}\, X$ for the set of equivalence relations on X, and write $\mathrm{Part}\, X$ for the set of partitions of X. Given $R \in \mathrm{Equiv}\, X$, let $\phi(R)$ denote the partition X/R defined in class. So, explicitly, X/R is the set of equivalence classes $[a]$ for the relation R of elements $a \in X$. Show that the resulting map $\phi : \mathrm{Equiv}\, X \to \mathrm{Part}\, X$ is one-one and onto. In other words, show that there is a one-one correspondence between equivalence relations on X and partitions of X.

2. (**5 points each**) For each nonnegative integer n, write $[n] := \{i \in \mathbb{Z} : 0 < i \leq n\}$. So $[n]$ is a set with n elements. (I apologize in advance that this is the same notation we use for equivalence classes.) Note that, by this definition, $[0] = \emptyset$.

Write P_n for the set of all partitions of the set $[n]$, and write $p(n)$ for the number of elements of P_n.

(1) Compute the integers $p(0), p(1)$ and $p(2)$.
(2) Write down the sets P_0, P_1 and P_2 explicitly. (Be careful about the placement and number of braces!)

(3) For $n > 1$, define a mapping $f : P_n \to 2^{[n-1]}$ as follows. If $\mathscr{P} = \{S_1, \dots, S_k\}$ is a partition of $[n]$ and S_i is the part of $\mathscr{P}$ containing n, then $f(\mathscr{P}) = S_i \cap [n-1]$. Show that the fiber of f over an element $T \in 2^{[n-1]}$ has $p(k)$ elements where $k = n - 1 - |T|$. (Here remember that elements of the power set are just subsets $T \subset [n-1]$.)

(4) Using (3), conclude that, for $n > 0$,

$$p(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} p(i).$$

(5) Using (4) and a computer compute $p(10)$. (You could use Sage for example.)

3. Suppose $f : X \to Y$ is a morphism in a category $\mathbf{C}$. If Z is an object in $\mathbf{C}$, then I write

$$f_* : \mathrm{Hom}_{\mathbf{C}}(Z, X) \to \mathrm{Hom}_{\mathbf{C}}(Z, Y)$$

for the map $g \mapsto fg$ and I write

$$f^* : \mathrm{Hom}_{\mathbf{C}}(Y, Z) \to \mathrm{Hom}_{\mathbf{C}}(X, Z)$$

for the map $g \mapsto gf$. Sometimes I call f_* the *pushforward* and f^* the *pullback*.

Show that the following are equivalent (**10 points**):

(a) f has a right inverse $h : Y \to X$.

(b) f_* is onto for all objects Z in $\mathbf{C}$.

Analogously, show that the following are equivalent (**10 points**):

(c) f has a left inverse $h : Y \to X$.

(d) f^* is onto for all objects Z in $\mathbf{C}$.

We say that f is a *split epimorphism* if f has a right inverse, and we say that f is a *split monomorphism* if f has a left inverse.

(**5 points**) Use Exercise I.4.3 from the text to explain why the above terminology makes sense. In other words, explain why split epimorphisms are epimorphisms and why split monomorphisms are monomorphisms. Also explain why a morphism which is both split mono and split epi is an isomoprhism. (*Mono* and *epi* are short for monomorphism and epimorphism.)

4. (**25 points**) Suppose $f : X \to Y$ is a morphism in a category $\mathbf{C}$. Show that the following are equivalent:

(1) f is an isomorphism.

(2) f is a monomorphism and a split epimorphism.