HW3, due Wednesday, September 22
Math 600, Fall 2023
Patrick Brosnan, Instructor

Practice Problems and Reading: Read Sections I.5 and II.1 of Aluffi's book. Then do the following problems from Aluffi for practice, but do not turn them in.

I.5) 6, 8, 11
II.1) 4, 5, 8, 13

The format is that "I.1" means the exercises from Section 1 of Chapter I in Aluffi's book (which start on page 8).

Terminology: Recall from class that if X is a set and T is a subset of $X \times X$, then the equivalence relation on X generated by T is the intersection of all equivalence relations on X containing T.

Graded Problems: Work the following problems for a grade. Turn them in on Canvas.

1. (25 points) Suppose X is a set and $R \subseteq X \times X$ is a relation on X which is reflexive and symmetric. Let $\overline{R}$ denote the equivalence relation on X generated by R. Suppose $a, b \in X$. Show that the following conditions are equivalent:

(1) $(a,b) \in \overline{R}$;
(2) there exists an integer $n \geq 1$ and elements $x_0, x_1, \ldots, x_n$ in X with $a = x_0$ and $b = x_n$ and with $(x_{i-1}, x_i) \in R$ for $i = 1, \ldots, n$.

2. (25 points) Suppose $\mathbf{C}$ is a category with a final object 1, and suppose X is an object in $\mathbf{C}$. Show that the product of X and 1 exists in $\mathbf{C}$. What is it?

3. An object $\mathbf{0}$ in category $\mathbf{C}$ is said to be a *zero object* if it both initial and final.

(1) **(13 points)** Show that the category $\mathrm{FVect}_{\mathbf{R}}$ of finite dimensional vector spaces over $\mathbf{R}$ has a zero object. What is it?
(2) **(12 points)** Show that the category of pointed sets (from Example 3.8) also has a zero object. What is it?

4. Suppose X is a set. If $U \subseteq V \subseteq X$, write $i_{U,V} : U \to V$ for the inclusion of U in V. In other words, $i_{U,V}(x) = x$ for $x \in U$.

Define a category $\mathbf{C}$ as follows:

- objects are subsets $U \subset X$;
- $\mathrm{Hom}_{\mathbf{C}}(U,V) = \begin{cases} i_{U,V}, & \text{if } U \subseteq V; \\ \emptyset, & \text{otherwise.} \end{cases}$
- composition of morphisms is composition of functions.

(1) **(13 points)** Show that if U and V are objects in **C**, then the product of U and V exists in **C**. What is it?

(2) **(12 points)** Show that if U and V are objects in **C**, then the coproduct of U and V exists in **C**. What is it?