HW4, due Wednesday, September 27
Math 600, Fall 2023
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Practice Problems and Reading: Read Sections II.2 and II.3 of Aluffi's book. Then do the following problems from Aluffi for practice, but do not turn them in.

II.2) 1, 2, 6
II.3) 1, 3

The format is that "I.1" means the exercises from Section 1 of Chapter I in Aluffi's book (which start on page 8).

Terminology: Recall from class that if X is a set and T is a subset of $X \times X$, then the equivalence relation on X generated by T is the intersection of all equivalence relations on X containing T.

Graded Problems: Work the following problems for a grade. Turn them in on Canvas.

1. Suppose $\mathbf{C}$ is a category and suppose $f : X \to Z$ and $g : Y \to Z$ are two morphisms in $\mathbf{C}$. We say that the *fiber product of X and Y over Z* exists and is equal to $(L, \alpha : L \to X, \beta : L \to Y)$, for L an object in $\mathbf{C}$, if $f\alpha = g\beta$ and $(f\alpha, \alpha, \beta)$ is the product of f with g in the category $\mathbf{C}_Z$ of objects in $\mathbf{C}$ over Z.

Suppose that the product $(M, \pi_1 : M \to X, \pi_2 : M \to Y)$ of X and Y exists in $\mathbf{C}$. Show that the fiber product of X and Y over Z exists if and only if the equalizer of the two maps $f\pi_1, g\pi_2 : M \to Z$ exists.

Hint: If $(E, h : E \to M)$ is the equalizer of of $f\pi_1$ and $g\pi_2$, show that $(E, \pi_1 h : E \to X, \pi_2 h : E \to Y)$ is the fiber product of X and Y over Z. And, if (L, α, β) is the fiber product of X and Y over Z, show that there's a map $\alpha \times \beta : L \to M$ such that $(L, \alpha \times \beta)$ is the equalizer of $f\pi_1$ and $g\pi_2$.

2. Show that, in Sets, the fiber product of any two morphism $f : X \to Z$ and $g : Y \to Z$ exists and is given by $X \times_Z Y$ from Problem 3 on HW1. (The maps $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ are the projections on the first and second factors.)

3. Suppose $f : M \to N$ is a magma isomorphism between monoids. Show that f is a monoid isomorphism.

4. Suppose $M = (M, \cdot)$ is a magma. The *opposite magma* is the magma $M^{\text{op}} = (M, *)$ where, for $x, y \in M$, $x * y = y \cdot x$.

(a) Show that M^{op} is a monoid if M is a monoid, and M^{op} is a group if M is.

(b) Show that, if M is a group, then M is always isomorphic as a group to M^{op}. (Hint: think of something you can do with group elements that reverses the order of multiplication.)

(c) Let $M = \text{End}_{\text{Sets}}\{1,2\}$. Show that M is not isomorphic (as a monoid, or even as a magma) to its opposite magma M^{op}.

(d) Let $M = M_n(\mathbb{R})$ denote the set of $n \times n$ matrices with real coefficients. Consider M as a monoid under matrix multiplication. Show that $M \cong M^{\text{op}}$. (Hint: think of something you can do with matrices that reverses the order of multiplication.)

5. Write $\mathbb{N} = \{0, 1, 2, \dots\}$ and regard it as a monoid under addition. If M is another monoid, we have a map

$$\text{ev} : \text{Hom}_{\text{Monoids}}(\mathbb{N}, M) \to M$$

given by $\text{ev}(\phi) = \phi(1)$. Show that ev is one-one and onto. (Hint: Let 1_M denote the identity element of M. Given $g \in M$, define a map $\phi_g : \mathbb{N} \to M$ inductively by setting $\phi_g(0) = 1_M$ and $\phi_g(n) = g\phi_g(n-1)$ for n a positive integer. Show that $\phi_g : \mathbb{N} \to M$ is a monoid homomorphism.)

The name ev comes from the word "evaluation." What it does is to evaluate a homomorphism ϕ at 1.