HW8, due Wednesday, November 8
Math 600, Fall 2023
Patrick Brosnan, Instructor

Practice Problems and Reading: Read Sections II.8 and III.1 of Aluffi's book. Work the following problems, but don't turn them in for a grade.

- II.8: 17
- III.1: 1, 6, 12

Graded Problems: Work the following problems for a grade. Turn them in on Canvas.

1. (20 points) Show that $\gcd(a,n) = \gcd(a+bn,n)$ for any three integers a, b and n.

2. (30 points) Suppose n is an integer. Show that

(a) $\mathrm{End}_{\mathrm{Groups}}(\mathbb{Z}/n,+) = (\mathbb{Z}/n,\times)$;
(b) $\mathrm{Aut}_{\mathrm{Groups}}(\mathbb{Z}/n,+) = (\mathbb{Z}/n,\times)^{\times}$;
(c) $(\mathbb{Z}/n,\times)^{\times} = \{[a] \in \mathbb{Z}/n : \gcd(a,n) = 1\}$.

Here the notation is as follows: $(\mathbb{Z}/n,+)$ is the group $\mathbb{Z}/n$ with the operation of addition; $(\mathbb{Z}/n,\times)$ is the monoid $\mathbb{Z}/n$ with the operation of multiplication; for any monoid M, $M^{\times}$ is the group of units of M.

Note that, by Problem 1, $[a] = [a'] \in \mathbb{Z}/n \Rightarrow \gcd(a,n) = \gcd(a',n)$.

In (a), the map from $(\mathbb{Z}/n,\times)$ to $\mathrm{End}_{\mathrm{Groups}}(\mathbb{Z}/n),+)$ is the one taking $x \in \mathbb{Z}/n$ to the map $\phi_x : \mathbb{Z}/n \to \mathbb{Z}/n$ given by $\phi_x(y) = xy$. For (a), you should check that,

(i) for each x, $\phi_x \in \mathrm{End}_{\mathrm{Groups}}(\mathbb{Z}/n),+)$,
(ii) the map $x \mapsto \phi_x$ is a monoid isomorphism from $(\mathbb{Z}/n,\times)$ to $\mathrm{End}_{\mathrm{Groups}}(\mathbb{Z}/n),+)$.

3. (20 points) Suppose $f : G \to Q$ is a split epimorphism in the category of groups. Show that $G \cong K \rtimes_{\theta} Q$ where $K = \ker f$ and $\theta : Q \to \mathrm{Aut}_{\mathrm{Groups}} K$ is an action of Q on K by group automorphisms. (**Hint:** Use the splitting to get a subgroup of G isomorphic to Q and an action of that subgroup by inner automorphisms on K.)

4. (30 points) As most of you know, there was a (bad) typo on the last problem on Exam 1. Consequently, I have to throw out that problem (and give everyone full credit for it). Here's the corrected problem, which I'd like you to do for this homework.

Suppose G is a group and X is a G-set. If $H \leq G$, we write $X^H := \{x \in X : \mathrm{Stab}_G(x) \geq H\}$. Now consider G/H as a G-set under the usual left action, and define a map

$$\mathrm{ev} : \mathrm{Hom}_{G\text{-Sets}}(G/H, X) \to X$$

by $\mathrm{ev}(\phi) = \phi(H)$.

(a) Show that the image of ev is contained X^H.
(b) Show that the image of ev is equal to X^H.
(c) Show that ev is one-to-one.