HW6, due Friday October 28
Math 403, Fall 2011
Patrick Brosnan, Instructor

Reading Assignment

Finish reading Chapter 2.

Writing Assignement (20 points each)

Problem 1. How many group homorphisms are there from $\mathbf{Z}/6$ to $\mathbf{Z}/15$? Prove your answer.

Problem 2. If G is a group, then a *group endomorphism* of G is a map $f : G \to G$ which is a group homomorphism. We write $\operatorname{End}_{\mathbf{Gps}} G$ for the set of all group endomorphisms or just $\operatorname{End} G$ when it is clear that we are talking about group endomorphisms. Now let $G = \mathbf{Z}/n$ is the cyclic group of order n (with addition as the binary operation). Show that every $\phi \in \operatorname{End} G$ is of the form $\phi([k]) = [m][k]$ for some $[m] \in \mathbf{Z}/n$.

Problem 3. Let G be a group and H an index 2 subgroup. Prove that H is normal in G.

Problem 4. Show that any group of order ≤ 5 is abelian, and that, up to isomorphism, the symmetric group on 3 letters is the only non-abelian group of order 6.

Problem 5. Let G be a group and $R \subset G \times G$ a subgroup that is also an equivalence relation. Set

$$N(R) := \{x \in G : (x, e) \in R\}.$$

Show that $N(R)$ is a normal subgroup of G.

Problem 6. Let G be a group. Write **Normal** for the set of all normal subgroups of G and **Equiv** for the set of all subgroups of $G \times G$ which are equivalence relations. Show that the map **Equiv** $\to$ **Normal** given by $R \mapsto N(R)$ is a 1-1 correspondence.

Problem 7. Let $G = S_3 = A(\{1, 2, 3\})$ denote the symmetric group on three symbols. Define a map $h : G \to \{\pm 1\}$ by $h(g) = (-1)^{o(g)+1}$. Show that h is a group homomorphism (where $\{\pm 1\}$ is a group under multiplication).

Problem 8. Let S be a set of points in $\mathbf{R}^2$ and let $\mathbf{GL}_2(\mathbf{R})$ denote the group of all invertible 2×2 matrices. For each $g \in \mathbf{GL}_2(\mathbf{R})$ let $gS = \{gs : s \in S\}$. Let $G = \{g \in \mathbf{GL}_2(\mathbf{R}) : gS = S\}$. Show that G is a subgroup of $\mathbf{GL}_2(\mathbf{R})$.

Problem 9. Let n be a integer with $n > 2$ and set $\theta = 2\pi/n$. For each integer k with $0 \leq k < n$ let

$$r_k = (\cos k\theta, \sin k\theta) \in \mathbf{R}^2.$$

Let $S = \{r_0, \ldots, r_{n-1}\}$, and let $G = \{g \in \mathbf{GL}_2(\mathbf{R}) : gS = S\}$. Set

$$X = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $X, Y \in G$, that $X^n = Y^2 = e$ and that $YXY^{-1} = X^{-1}$.

Problem 10. Show that the subgroup of $\mathbf{GL}_2$ generated by X and Y as in Problem 9 has exactly $2n$ elements. This group is called the *dihedral group* of order $2n$.

Bonus. (10 points) Show that G is the subgroup of $\mathbf{GL}_2(\mathbf{R})$ generated by X and Y.