HW8, due Friday, November 11
Math 403, Fall 2011
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Reading Assignment

Begin reading about rings in Chapter 3.1-2.

Problem 1. (20 points) Suppose S is a set and G is a group. Let G^S denote the set of all functions $f : S \to G$. Define a binary operation $*$ on G^S by setting $(f * g)(s) = f(s)g(s)$. Show that G^S with the binary operation $*$ is a group. Moreover, show that G^S is abelian if G is.

Problem 2. (20 points) Suppose G is a group and A is an abelian group.
(a) Show that the set $\mathrm{Hom}(G, A)$ of all group homomorphisms from G to A is a subgroup of A^G.
(b) If $f : G \to H$ is a homomorphism of groups, define $f^* : \mathrm{Hom}(H, A) \to \mathrm{Hom}(G, A)$ by $f^*(\gamma) = \gamma \circ f$. Show that f^* is a group homomorphism.
(c) Suppose $g : H \to K$ is another group homomorphism, so that (b) gives us a group homomophisms $g^* : \mathrm{Hom}(K, A) \to \mathrm{Hom}(H, A)$. Show that $(g \circ f)^* = f^* \circ g^*$.

Problem 3. (20 points) Let $\mathbf{Q}$ denote the group of rational numbers (with addition as the binary operation) and let $\mathbf{Z}$ denote the subgroup of integers. The *Pontryagin dual* of a group G is the group $G^* = \mathrm{Hom}(G, \mathbf{Q}/\mathbf{Z})$. (It is most useful when G is abelian).
(a) Show that if $G = H \times K$, the G^* is isomorphic to $H^* \times K^*$.
(b) Show that G^* is finite if G is a finite group.
(c) Show that, if n is a non-negative integer, then $(\mathbf{Z}/n)^*$ is isomorphic to $\mathbf{Z}/n$.
(d) Suppose $g \in G$. Define a map $\mathrm{ev}_g : G^* \to \mathbf{Q}/\mathbf{Z}$ by $\mathrm{ev}_g(\lambda) = \lambda(g)$. Show that $\mathrm{ev}_g \in \mathrm{Hom}(G^*, \mathbf{Q}/\mathbf{Z}) = G^{**}$.
(e) Define a map $\mathrm{ev} : G \to G^{**}$ by $g \mapsto \mathrm{ev}_g$. Show that ev is a group homomorphism.

Problem 4. (20 points) Suppose $G = \mathbf{Z}/n$ for some non-negative integer n. Show that G^* is isomorphic to $\mathbf{Z}/n$. Compute G^* (up to isomorphism) for the symmetric group on 3 letters.

Problem 5. Find an example of a non-abelian group G such that every subgroup H of G is normal. (**Hint:** Look for one of order ≤ 8.)