HW5, due Wednesday, March 26
Math 403, Spring 2014
Patrick Brosnan, Instructor

1. Suppose $\phi : G \to H$ is a surjective group homomorphism and $|G| < \infty$. Show that the order of H divides the order of G.

2. Suppose G is a finite group and N is a normal subgroup. Write $\pi : G \to G/N$ for the map given by $g \mapsto gN$. Show that, for $g \in G$, $|\pi(g)|$ divides $|g|$.

3. Suppose G is a group and $H, K \leq G$. We say that H and K are *conjugate* in G if there exists a $g \in G$ such that $H = gKg^{-1}$. Write $H \sim K$ if H and K are conjugate.

(1) Show that $H \sim H$, that $H \sim K$ implies $K \sim H$ and that $H_1 \sim H_2$ and $H_2 \sim H_3$ implies that $H_1 \sim H_3$. In other words, show that $\sim$ is an equivalence relation on the set of subgroups of G.
(2) Show that conjugate subgroups are isomorphic.

4. Suppose G is a finite subgroup of $\mathbf{O}_2(\mathbb{R})$. Show that G is conjugate to a subgroup of $\mathbf{D}_n$ for some n.

5. Suppose G is a finite abelian group and p is a prime dividing the order of G.

(1) Show that, if G is cyclic, then G contains an element of order p.
(2) Suppose H is a subgroup of G and suppose G/H contains an element of order p. Show that G contains an element of order p.
(3) Using induction on the order of G, show that every finite abelian group of order divisible by p contains an element of order p.