HW1, due Wednesday, April 9
Math 403, Spring 2014
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Recall from class that an action of a group G on a set X is group homomorphism $\rho : G \to A(X)$. A pair (X,ρ) where X is a group and $\rho : G \to A(X)$ is a group homomorphism is called a G-set. If (X,ρ) is a G-set and if $g \in G, x \in X$, then we write gx for $\rho(g)(x)$. The stabilizer of an element $x \in X$ is the subgroup $G_x = \{g \in G : gx = x\}$ and the orbit of x is Gx. A G-set is said to be transitive if it has exactly one orbit.

Suppose G is a group. One example of a group action is the action of $G \times G$ on G given by $\rho(x,y)(g) = xgy^{-1}$.

1. Suppose G is a group and $\rho : G \times G \to A(G)$ is the group action defined above. Show that the stabilizer of an element $g \in G$ is the set

$$\{(ghg^{-1},h) : h \in G\}.$$

2. Suppose $\rho : G \to \mathbb{Z}/10\mathbb{Z}$ is a surjective group homomorphism Show that G has normal subgroups of index $2, 5$ and 10.

3. Let $\mathbb{C}$ denote the complex numbers. So $\mathbb{C} = \{x+iy : x,y \in \mathbb{R}\}$ where i denotes the imaginary number $\sqrt{-1}$. Let $G = \mathbf{GL}_2(\mathbb{C})$ denote the set of 2×2-matrices with entries in $\mathbb{C}$. It is easy to see that G is a group under matrix-multiplication. Let Q denote the subgroup of G generated by the matrices

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix};$$
$$Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Write Id for the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(1) Show that $X^2 = Y^2 = (XY)^2 = -\mathrm{Id}$.
(2) Show $XY = -YX$.
(3) Set $Z = XY$, and show that $Q = \{\pm\mathrm{Id}, \pm X, \pm Y, \pm Z\}$ so that Q has 8 elements.
(4) Show that Q is non-abelian and that Q is not isomorphic to $\mathbf{D}_4$.

4. Suppose G is a group. Recall that the commutator of two elements $x,y \in G$ is the element $[x,y] = xyx^{-1}y^{-1}$ of G. The commutator of G is the subgroup $[G,G]$ of G generated by all commutators $[x,y]$ with $x,y \in G$.

(1) Show that $[G,G] \trianglelefteq G$.
(2) Suppose $\rho : G \to H$ is a group homomorphism and H is an abelian group. Show that $[G,G] \le \ker\rho$.
(3) Define $G_{\mathrm{ab}} := G/[G,G]$. Show that G_{ab} is abelian. G_{ab} is called the abelianization of G.
(4) Suppose $G = \mathbf{D}_n$ is the dihedral group and $n > 2$. Show that $G_{\mathrm{ab}} \cong \mathbf{C}_2$ for n odd and $G_{\mathrm{ab}} \cong \mathbf{C}_2 \times \mathbf{C}_2$ for n even.

5. Suppose G is a group and n is a positive integer. Suppose $g \in G$ is the unique element of order n. Show that g is in the center of G. (That is $g \in Z(G)$.)