HW1, due Friday, February 5
Math 404, Spring 2014
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1. Let $\mathbb{Q}$ denote the group of rational numbers (with addition as the binary operation).

(a) Show that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.
(b) Show that $\mathbb{Q}$ itself is not finitely generated.

2. Suppose n and m are positive integers. Let $a = \gcd(n,m)$, $b = \operatorname{lcm}(n,m)$, and let x and y be integers such that $xn + ym = a$. Set

$$M := \mathbb{Z}/n \times \mathbb{Z}/m,$$

and let H and K denote the cyclic subgroups generated by $(1,1)$ and $(xn/a, -ym/a)$ respectively. Prove the following:

(a) H is cyclic of order b.
(b) H is in the kernel of a group homomorphism $\varphi : M \to \mathbb{Z}/a$ sending (u,v) to $u - v \pmod{a}$.
(c) The restriction of φ to K induces an isomorphism from K to $\mathbb{Z}/a$.
(d) The group M is the direct product of H with K. Conclude that

$$M \cong \mathbb{Z}/a \times \mathbb{Z}/b.$$

3. Suppose M is finite abelian group. Let r be the smallest possible number of elements in a basis for M and let $S = \{x_1, \dots, x_r\}$ be a basis for M such that $|x_1|$ is as small as possible. Set $d_i = |x_i|$ and show that $d_1 | d_i$ for all i. (Hint: Use Problem 2.)

4. Use Problem 3 and induction to show that every finite abelian group is isomorphic to a group of the form

$$\mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_r$$

where the d_i are positive integers and $d_1 | d_2 | \cdots d_r$.