HW5, due Wednesday, April 1
Math 404, Spring 2015
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1. We say that an angle θ is constructible if, starting from the points $\{(0,0),(1,0)\}$, it is possible to construct two lines intersecting at the origin and making the angle θ. Give an example of an angle θ such that θ is constructible but $\theta/5$ is not. (**Hint:** Howework 5 might be useful for this.)

2. The purpose of this problem and the next is to give a detailed proof of some results on field automorphisms which I went over in class (while I was answering a question).

Suppose F is a field and $f \in F[x]$ is a non-zero polynomial. Let E/F be a field extension generated by the set $S := \{\alpha_1,\dots,\alpha_s\}$ of roots of f in E. Let $\tau : F \to \Omega$ be an embedding of F in a field Ω, and let $T = \{\beta_1,\dots,\beta_t\}$ denote the set of roots of f in Ω. Recall that $\mathrm{Hom}_\tau(E,\Omega)$ denote the set of all extensions $\sigma : E \to \Omega$ of τ to E. Prove the following.

(1) For every $\sigma \in \mathrm{Hom}_\tau(E,\Omega)$, $\sigma(S) \subset T$.
(2) If $\sigma, \sigma' \in \mathrm{Hom}_\tau(E,\Omega)$ and $\sigma(\alpha_i) = \sigma'(\alpha_i)$ for all i, then $\sigma = \sigma'$.

3. Suppose F is a field and $f \in F[x]$ is a non-zero polynomial. As in the previous problem, let E/F be a field extension generated by the set $S = \{\alpha_1,\dots,\alpha_s\}$ of roots of f in E. Let $\mathrm{Aut}_F E$ denote the set of all F-linear field automorphisms of E, and let $\mathrm{Aut}\, S$ denote the set of all automorphism of the set S. Both $\mathrm{Aut}_F E$ and $\mathrm{Aut}\, S$ are groups with composition as the group operation. For each $\sigma \in \mathrm{Aut}_F E$, let $\rho(\sigma) = \sigma_{|S}$ denote the restriction of σ to S. Prove the following:

(1) For each $\sigma \in \mathrm{Aut}_F E$, $\rho(\sigma) \in \mathrm{Aut}\, S$.
(2) The map $\rho : \mathrm{Aut}_F E \to \mathrm{Aut}\, S$ is a group homomorphism.
(3) The kernel of ρ is trivial.

4. Let $f = x^3 - 2 \in \mathbb{Q}[x]$, and let E denote the splitting field of f. Let $L = \mathbb{Q}(2^{1/3}) \subset E$ and let $M = \mathbb{Q}(e^{2\pi i/3}) \subset E$. Say what the following groups are up to isomorphism.

(1) $\mathrm{Aut}_{\mathbb{Q}} E$.
(2) $\mathrm{Aut}_{\mathbb{Q}} L$.
(3) $\mathrm{Aut}_{\mathbb{Q}} M$.
(4) $\mathrm{Aut}_L E$.
(5) $\mathrm{Aut}_M E$.

5. Do Problem 2-1 on page 33 of Milne's Field Theory book.