HW8, due Wednesday, April 29
Math 404, Spring 2015
Patrick Brosnan, Instructor

1. Let $G = \mathbf{GL}_2(\mathbb{R})$ denote the group of invertible real 2×2 matrices, and let $X = \mathbb{R}^2$. Let G act on X by matrix multiplication.

(a) What are the orbits of G acting on X?
(b) For each orbit Y, pick $v \in Y$ and say what the stabilizer is.

2. Suppose Y is a G-set and X is a transitive G set. Let $x \in X$ be a point with stabilizer subgroup H so that X is isomorphic as a G-set to G/H. Set $Y^H := \{y \in Y : H \leq G_y\}$.

(1) Suppose $f : X \to Y$ is a morphism of G-sets. Show that $f(x) \in Y^H$.
(2) Conversely, show that, for every $y \in Y^H$, there exists exactly one morphism $f : X \to Y$ of G-sets such that $f(x) = y$.

3. Suppose E/F is a finite Galois extension with Galois group G, and let L be another field extension of F. Let $X := \mathrm{Hom}_F(L,E)$ denotes the set of all F-linear embeddings of L into E.

(a) For each $\sigma \in X$ and $g \in G$, show that $g \circ \sigma$ is also in X.
(b) Show that the map $G \times X \to X$ given by $(g,\sigma) \mapsto g\circ\sigma$ defines an action of G on X.
(c) Suppose X is non-empty, so that there exists an F-linear embedding $\sigma : L \to E$. Set $H = \mathrm{Gal}(E/\sigma(L))$. Show that X is transitive and that the stabilizer of σ is H.

4. Suppose E/F is a finite Galois extension with Galois group G, and let L and M denote two other extensions of F. Set $X = \mathrm{Hom}_F(L,E)$ and $Y = \mathrm{Hom}_F(M,E)$. View these as G-sets as in Problem 2.

(a) Suppose $\varphi : L \to M$ is an F-linear embedding. Show that the map $\varphi^* : Y \to X$ given by $\sigma \mapsto \sigma\circ\varphi$ is a morphism of G-sets.
(b) **(10 point Bonus)** Suppose $f : Y \to X$ is a morphism of G-sets and that Y is non-empty, so that there is an F-linear embedding $\sigma : M \to E$. Show that there is an embedding of field $\varphi : L \to M$ such that $f = \varphi^*$.

5. Suppose E/F is a Galois extension with Galois group G and L is an intermediate field with $H = \mathrm{Gal}(E/L)$. Let N denote the normalizer of H in G. Show that the group $\mathrm{Aut}_F L$ of F-linear automorphisms of L is isomorphic to N/H.