Dot Product, Cross Product, Determinants

We considered vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. We will write $\mathbb{R}^d$ for statements which work for $d = 2,3$ (and actually also for $d = 4,5,\ldots$ although this is not needed in this course).

Our goal is to measure lengths, angles, areas and volumes. This will be used later for lengths of curves, surface areas, volumes with curved surfaces etc.

Review of Dot Product

For vectors $\vec{a},\vec{b} \in \mathbb{R}^d$ we define the **dot product** by

$$\vec{a}\cdot\vec{b} = a_1b_1 + \cdots + a_db_d.$$

The **length** or **norm** of a vector $\vec{a} \in \mathbb{R}^d$ is defined by

$$\|\vec{a}\| := (\vec{a}\cdot\vec{a})^{1/2} = \sqrt{a_1^2 + \cdots + a_d^2}.$$

There holds

$$\left|\vec{a}\cdot\vec{b}\right| \le \|\vec{a}\| \left\|\vec{b}\right\| \qquad \text{Cauchy-Schwarz inequality}$$
$$\left\|\vec{a}+\vec{b}\right\| \le \|\vec{a}\| + \left\|\vec{b}\right\| \qquad \text{triangle inequality}$$

If we denote the angle between nonzero vectors $\vec{a},\vec{b}$ by θ there holds

$$\vec{a}\cdot\vec{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta$$

This can be used to compute the **angle θ between two vectors** $\vec{a},\vec{b}$ by

$$\boxed{Q := \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\| \left\|\vec{b}\right\|}, \qquad \theta = \cos^{-1} Q.}$$

By the Cauchy-Schwarz inequality we have $-1 \le Q \le 1$, so applying the inverse cosine gives a value $\theta \in [0,\pi]$.

Two vectors $\vec{a},\vec{b}$ are **parallel** if there exists a scalar $q \in \mathbb{R}$ such that $\vec{b} = q\vec{a}$. If $q > 0$ we have $Q = 1$ and $\theta = 0$. If $q < 0$ we have $Q = -1$ and $\theta = \pi$.

Two vectors $\vec{a},\vec{b}$ are **orthogonal** if $\vec{a}\cdot\vec{b} = 0$. In this case we have $Q = 0$ and $\theta = \pi/2$.

Let $\vec{a},\vec{b} \in \mathbb{R}^d$ with $\vec{a} \neq \vec{0}$. We want to decompose the vector $\vec{b}$ into a vector parallel to $\vec{a}$ and a vector orthogonal to $\vec{a}$:

$$\vec{b} = \vec{c} + \vec{d}, \qquad \vec{c} \parallel \vec{a}, \quad \vec{d} \perp \vec{a} \tag{1}$$

We must have $\vec{c} = q\vec{a}$ where $q \in \mathbb{R}$ is chosen such that $\vec{d} = \vec{b} - q\vec{a} \perp \vec{a}$, i.e.,

$$(\vec{b} - q\vec{a})\cdot\vec{a} = 0 \iff \vec{b}\cdot\vec{a} - q\vec{a}\cdot\vec{a} = 0 \iff q = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}$$

We define the **projection of the vector $\vec{b}$ onto the vector $\vec{a}$** as this vector $\vec{c}$, i.e.,

$$\boxed{\mathrm{pr}_{\vec{a}}\vec{b} := \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}}$$

Note that $\left\|\mathrm{pr}_{\vec{a}}\vec{b}\right\| = \dfrac{\left|\vec{a}\cdot\vec{b}\right|}{\|\vec{a}\|}$.

Area of a Parallelogram

For vectors $\vec{a},\vec{b} \in \mathbb{R}^d$ we now consider the parallelogram with the four vertices $\vec{0},\vec{a},\vec{b},\vec{a}+\vec{b}$. The area A of this parallelogram can be computed as base times height.

If the angle between the vectors $\vec{a},\vec{b}$ is denoted by θ we obtain the area as

$$A = \|\vec{a}\|\,\|\vec{b}\|\sin\theta. \tag{2}$$

If we decompose $\vec{b} = \vec{c}+\vec{d}$ as in (1) we can write the area as

$$A = \|\vec{a}\|\,\|\vec{d}\| = \|\vec{a}\|\,\|\vec{b} - \mathrm{pr}_{\vec{a}}\vec{b}\|$$

Since $\vec{c} \perp \vec{d}$ we have $\|\vec{d}\|^2 = \|\vec{b}\|^2 - \|\vec{c}\|^2$ and obtain

$$A^2 = \|\vec{a}\|^2 \left(\|\vec{b}\|^2 - \|\mathrm{pr}_{\vec{a}}\vec{b}\|^2\right) = \|\vec{a}\|^2 \left(\|\vec{b}\|^2 - \frac{(\vec{a}\cdot\vec{b})^2}{\|\vec{a}\|^2}\right)$$

$$\boxed{A^2 = \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2} \tag{3}$$

In order to write A in a different way we consider

$$\sum_{i=1}^{d}\sum_{j=1}^{d}(a_ib_j - a_jb_i)^2 = \left(\sum_{i=1}^{d} a_i^2\right)\left(\sum_{j=1}^{d} b_j^2\right) - 2\left(\sum_{i=1}^{d} a_ib_i\right)\left(\sum_{j=1}^{d} a_jb_j\right) + \left(\sum_{j=1}^{d} a_j^2\right)\left(\sum_{i=1}^{d} b_i^2\right)$$
$$= \|\vec{a}\|^2\|\vec{b}\|^2 - 2(\vec{a}\cdot\vec{b})^2 + \|\vec{a}\|^2\|\vec{b}\|^2 = 2A^2$$

Note that $(a_ib_j - a_jb_i)^2$ is zero for $i = j$, and that its value is the same if we interchange i and j. Hence we obtain

$$\boxed{A^2 = \sum_{\substack{i,j=1,\dots,d\\ i<j}} (a_ib_j - a_jb_i)^2} \tag{4}$$

For $d = 2$ this sum has one term, for $d = 3$ this sum has 3 terms (for $d = 4$ it has 6 terms). We will now look at this formula for $d = 2$ and $d = 3$.

Parallelogram in $\mathbb{R}^2$

For $d = 2$ the sum in (4) contains only one term with $i = 1$, $j = 2$ and we have

$$A^2 = (a_1b_2 - a_2b_1)^2$$
$$A = |a_1b_2 - a_2b_1|\,. \tag{5}$$

Note that the expression $a_1b_2 - a_2b_1$ is the so-called **determinant of the 2 by 2 matrix** consisting of the vectors $\vec{a},\vec{b}$:

$$\boxed{\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} := a_1b_2 - a_2b_1}$$

The determinant may be negative. Its absolute value is the area of the parallelogram:

$$\boxed{A = \left|\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right|}.$$

The sign of the determinant specifies the **orientation** of two vectors $\vec{a},\vec{b} \in \mathbb{R}^2$: A positive determinant means that the vector $\vec{b}$ is to the left of vector $\vec{a}$, a negative determinant means that the vector $\vec{b}$ is to the right of vector $\vec{a}$.

Parallelogram in $\mathbb{R}^3$

For $d = 3$ the sum in (4) contains three terms and we have

$$A^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2.$$

Hence $A = \|\vec{x}\|$ with the vector $\vec{x}$ defined by

$$\vec{x} := (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

Note that $x_j = a_k b_\ell - a_\ell b_k$ where

1. j, k, ℓ are different
2. k, ℓ are in "positive order" if we arrange 1,2,3 on a circle.

This vector $\vec{x}$ is the so-called **cross product** of the vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$

$$\boxed{\vec{a} \times \vec{b} := (a_2b_3 - a_3b_2, a_3b_1 - a_3b_1, a_1b_2 - a_2b_1)}$$

Note that $\vec{x} := \vec{a} \times \vec{b}$ satisfies

$$\vec{a} \cdot \vec{x} = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0$$
$$\vec{b} \cdot \vec{x} = b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1) = 0.$$

The cross product $\vec{a} \times \vec{b}$ therefore has the following properties:

1. $\|\vec{x}\|$ is the area A of the parallelogram defined by $\vec{a}, \vec{b}$, i.e., $\|\vec{x}\| = \left\|\vec{a}\right\| \left\|\vec{b}\right\| \sin\theta$.
2. $\vec{x}$ is orthogonal to $\vec{a}$, $\vec{b}$.

Note that for two nonzero, non-parallel vectors $\vec{a}, \vec{b}$ there are two vectors $\vec{x}$ which satisfy these two conditions where one is the negative of the other. We will see below that the three vectors $\vec{a}, \vec{b}, \vec{x}$ should have "*positive orientation*" in the sense of the "**right hand rule**": If the thumb of the right hand points in the direction of $\vec{a}$, the index finger in the direction of $\vec{b}$, then $\vec{x}$ points in the direction given by the middle finger. This resolves the ambiguity between the two possible vectors.

Area of Triangle

For vectors $\vec{a}, \vec{b} \in \mathbb{R}^d$ we consider the triangle with the three vertices $\vec{0}, \vec{a}, \vec{b}$. Obviously the area of this triangle is $\frac{1}{2}A$ where A is the area of the parallelogram with vertices $\vec{0}, \vec{a}, \vec{b}, \vec{a} + \vec{b}$.

Volume of Parallelepiped in $\mathbb{R}^3$

Three vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ define a so-called *parallepiped*. This is a polyhedron with 6 sides which are parallelograms. Its 8 vertices are given by the points $r\vec{a} + s\vec{b} + t\vec{c}$ where $r, s, t \in \{0, 1\}$. Its volume V is computed as (area of base) times height.

If we consider the parallelogram given by the vectors $\vec{a}, \vec{b}$ as the base, its area is $\|\vec{x}\|$ where $\vec{x} := \vec{a} \times \vec{b}$.

Note that the vector $\vec{x}$ is orthogonal on the parallelogram. Therefore the height h is the component of the vector $\vec{c}$ in the direction of $\vec{x}$, i.e.,

$$h = \|\mathrm{pr}_{\vec{x}}\,\vec{c}\| = \frac{|\vec{c} \cdot \vec{x}|}{\|\vec{x}\|}$$

yielding for the volume V of the parallelepiped

$$V = \|\vec{x}\| \frac{|\vec{c} \cdot \vec{x}|}{\|\vec{x}\|} = |\vec{c} \cdot \vec{x}|$$
$$V = \left|(\vec{a} \times \vec{b}) \cdot \vec{c}\right|. \tag{6}$$

The expression $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is called the **determinant of the 3 by 3 matrix** consisting of the vectors $\vec{a}, \vec{b}, \vec{c}$:

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} := (\vec{a} \times \vec{b}) \cdot \vec{c}$$
$$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3$$

Note that this is the sum of terms $\pm a_j b_k c_\ell$ where

1. j, k, l are different
2. "+" if j, k, ℓ are in "positive order" , "−" if j, k, ℓ are in negative order

if we arrange 1,2,3 on a circle.

The absolute value of the determinant gives the volume V of the parallepiped. The sign of the determinant gives the **orientation** of the three vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$: A positive sign means that the vectors are arranged according to the "right hand rule", a negative sign means that the vectors are arranged according to the "left hand rule".

Note that for the three vectors $\vec{a}, \vec{b}, \vec{x} := \vec{a} \times \vec{b}$ we have

$$\det \begin{bmatrix} \vec{a} \\ \vec{b} \\ \vec{x} \end{bmatrix} = (\vec{a} \times \vec{b}) \cdot \vec{x} = \vec{x} \cdot \vec{x} \geq 0,$$

i.e., the three vectors $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ are positively oriented (if $\vec{a} \parallel \vec{b}$ we have $\vec{a} \times \vec{b} = \vec{0}$).

Volume of Tetrahedron in $\mathbb{R}^3$

For vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ we consider the tetrahedron with the four vertices $\vec{0}, \vec{a}, \vec{b}, \vec{c}$. We can compute the volume of a tetrahedron as (area of base) $\cdot$ height/3. Compared with the parallepiped, the area of the base is half as large. The volume V of the parallepiped was given as $V =$ (area of base) $\cdot$ height. Hence the volume of the tetrahedron is $\frac{1}{6}V$.