## Curves: Length, Tangent and Normal Vector, Curvature

A curve is given by a parametrization

$$\mathbf{r}(t) = (x(t), y(t), z(t)), \quad a \le t \le b$$

The **velocity** is $\mathbf{v}(t) = \mathbf{r}'(t)$, the **speed** is $V(t) = \|\mathbf{v}(t)\|$, the **acceleration** is $\mathbf{a}(t) = \mathbf{r}''(t)$.

The **length** $L$ **of the curve** is given by the integral over the speed

$$L = \int_a^b V(t)dt,$$

the **arc length** is given by $s(t) = \displaystyle\int_a^t V(u)du$ so that $\dfrac{ds}{dt} = V(t)$.

The velocity vector $\mathbf{v}$ is tangential to the curve at the point $\mathbf{r}(t)$. The **unit tangent vector T** is defined by

$$\mathbf{T} = \mathbf{v}/V. \tag{1}$$

We want to consider the function $\mathbf{T}(s)$ which gives the unit tangent vector for a point with arc length $s$, i.e.,

$$\begin{aligned} \mathbf{T}(s(t)) &= \mathbf{v}(t)/V(t) \\ \mathbf{v}(t) &= V(t)\mathbf{T}(s(t)) \end{aligned}$$

We take the derivative of this equation and obtain for $\mathbf{a}(t) = \mathbf{v}'(t)$ with the product and chain rule

$$\begin{aligned} \mathbf{a}(t) &= V'(t)\mathbf{T}(s(t)) + V(t)\mathbf{T}'(s(t))\underbrace{s'(t)}_{V(t)} \\ \mathbf{a}(t) &= \underbrace{V'(t)\mathbf{T}(s(t))}_{\mathbf{a}_{\text{par}}} + \underbrace{V(t)^2\mathbf{T}'(s(t))}_{\mathbf{a}_{\text{orth}}} \end{aligned}$$

Note that $\mathbf{T}(s)\cdot\mathbf{T}(s) = 1$ implies by differentiation $2\mathbf{T}'(s)\cdot\mathbf{T}(s) = 0$. Hence the vector $\mathbf{T}'(s)$ is orthogonal on the tangent vector $\mathbf{T}$. Therefore we have obtained a decomposition $\mathbf{a} = \mathbf{a}_{\text{par}} + \mathbf{a}_{\text{orth}}$ where $\mathbf{a}_{\text{par}}$ is parallel to $\mathbf{v}$ and $\mathbf{a}_{\text{orth}}$ is orthogonal on $\mathbf{v}$.

The length of $\mathbf{T}'(s)$ tells us about the change of the tangent vector as we move along the curve with speed 1, we define this as the **curvature** $\kappa$:

$$\kappa := \left\|\mathbf{T}'(s)\right\|$$

The **normal vector N** is defined as the unit vector in the direction of $\mathbf{T}'(s)$:

$$\mathbf{N} = \mathbf{T}'(s)/\left\|\mathbf{T}'(s)\right\|. \tag{2}$$

We therefore have with unit vectors $\mathbf{T}$, $\mathbf{N}$ the decomposition

$$\boxed{\mathbf{a} = V'\mathbf{T} + V^2\kappa\mathbf{N}}$$

which tells us that the acceleration vector is decomposed into

- a component parallel to the curve with size $V'(t)$, i.e., the change of speed
- a component orthogonal to the curve with size $V^2\kappa$, as consequence of the curvature

Recall the case of motion with constant speed $V$ around a circle $R$. In this case we obtained an acceleration of size $V^2\kappa$ with the curvature $\kappa = 1/R$.

We can find the decomposition $\mathbf{a} = \mathbf{a}_{\text{par}} + \mathbf{a}_{\text{orth}}$ (where $\mathbf{a}_{\text{par}}$ is parallel to $\mathbf{v}$ and $\mathbf{a}_{\text{orth}}$ is orthogonal on $\mathbf{v}$) as follows:

$$\mathbf{a}_{\text{par}} = \text{pr}_{\mathbf{v}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}, \qquad \mathbf{a}_{\text{orth}} = \mathbf{a} - \mathbf{a}_{\text{par}} \tag{3}$$

We have $\mathbf{a}_{\text{par}} = a_T \mathbf{T}$ with

$$a_T = V' = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|}. \tag{4}$$

We have $\mathbf{a}_{\text{orth}} = a_N \mathbf{N}$ with

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}. \tag{5}$$

The **curvature** $\kappa$ can then be computed as

$$\kappa = \frac{a_N}{V^2} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{V^3}. \tag{6}$$

The **binormal vector** $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is a unit vector which is orthogonal on $\mathbf{v}(t)$ and $\mathbf{a}(t)$. Hence we can compute it as

$$\mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{\|\mathbf{v} \times \mathbf{a}\|}$$

For computing $a_T, a_N, \kappa, \mathbf{T}, \mathbf{N}$ you should

- find the vectors $\mathbf{v}, \mathbf{a}$
- find $\mathbf{v} \cdot \mathbf{v}$, $\mathbf{v} \cdot \mathbf{a}$, $\mathbf{a} \cdot \mathbf{a}$ from which you get $a_T, a_N, \kappa$ by (4), (5), (6)
- find $\mathbf{T}$ using (1), find $\mathbf{N}$ using (3) and $\mathbf{N} = \mathbf{a}_{\text{orth}} / \|\mathbf{a}_{\text{orth}}\|$

If you only need $a_T(t_0), a_N(t_0), \kappa(t_0), \mathbf{T}(t_0), \mathbf{N}(t_0)$ for a given number $t_0$: First compute the two vectors $\mathbf{v}(t_0), \mathbf{a}(t_0)$. These vectors just contain numbers (without any $t$), and you can do all computations using these two vectors. That's how you should solve problem 3 below.

## Problem 1

Let $\mathbf{r}(t) = (3t, 4t^{3/2}, -3t^2)$ for $1 \le t \le 3$. Find the length of the curve.

## Problem 2

Let $\mathbf{r}(t) = (t^2, t, -t)$. Find the curvature $\kappa(t)$.

## Problem 3

Let $\mathbf{r}(t) = (t, t^2, t^3/3)$. For $t_0 = 1$ compute $\mathbf{N}$ and $\kappa$.