Solutions of Problems for Curves

Problem 1

Let $\mathbf{r}(t) = (3t, 4t^{3/2}, -3t^2)$ for $1 \le t \le 3$. Find the length of the curve.

Solution: $\mathbf{v}(t) = (3, 6t^{1/2}, -6t)$, speed is $\|\mathbf{v}(t)\| = \sqrt{9+36t+36t^2} = \sqrt{(3+6t)^2} = |3+6t|$, length is

$$L = \int_1^3 \|\mathbf{v}(t)\|\,dt = \int_1^3 |3+6t|\,dt = \int_1^3 (3+6t)dt = \left[3t+3t^2\right]_1^3 = 36-6 = 30.$$

Problem 2

Let $\mathbf{r}(t) = (t^2, t, -t)$. Find the curvature $\kappa(t)$.

Solution: $\mathbf{v}(t) = (2t, 1, -1)$, $\mathbf{a}(t) = (2,0,0)$. We have $\mathbf{v}(t)\cdot\mathbf{v}(t) = 4t^2+2$, $\mathbf{v}(t)\cdot\mathbf{a}(t) = 4t$, $\mathbf{a}(t)\cdot\mathbf{a}(t) = 4$ and hence

$$a_T = \frac{4t}{\sqrt{4t^2+2}}, \quad a_N = \sqrt{4 - \frac{(4t)^2}{4t^2+2}} = \sqrt{\frac{8}{4t^2+2}}, \quad \kappa = \frac{a_N}{\|\mathbf{v}\|^2} = \sqrt{\frac{8}{(4t^2+2)^3}} = \frac{1}{(2t^2+1)^{3/2}}.$$

Problem 3

Let $\mathbf{r}(t) = (t, t^2, t^3/3)$. For $t_0 = 1$ compute $\mathbf{N}$ and κ.

Solution: Here $\mathbf{v}(t) = (1, 2t, t^2)$, $\mathbf{a}(t) = (0,2,2t)$ and we evaluate these vectors at $t_0 = 1$: $\mathbf{v}(t_0) = (1,2,1)$, $\mathbf{a}(t_0) = (0,2,2)$. We can obtain all results from these two vectors $\mathbf{v} = (1,2,1)$ and $\mathbf{a} = (0,2,2)$. We compute

$$\mathbf{a}_{\text{par}} = \text{pr}_{\mathbf{v}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{6}{6}(1,2,1), \qquad \mathbf{a}_{\text{orth}} = \mathbf{a} - \mathbf{a}_{\text{par}} = (0,2,2) - (1,2,1) = (-1,0,1)$$

$$\mathbf{N} = \mathbf{a}_{\text{orth}} / \|\mathbf{a}_{\text{orth}}\| = \frac{1}{\sqrt{2}}(-1,0,1)$$

$$a_N = \|\mathbf{a}_{\text{orth}}\| = \sqrt{2}, \qquad \kappa = \frac{a_N}{\mathbf{v}\cdot\mathbf{v}} = \frac{\sqrt{2}}{6}$$