Math 241, Spring 14
Practice Problems for Exam 3

1.

(a) Let R denote the region of points $(x,y) \in \mathbb{R}^2$ such that $0 \le y \le 4-x^2$. Find the integral $\iint_{(x,y)\in R} (x^2+y)dA$.

(b) Change the order of integration in the following double integral:

$$\int_{x=0}^{2}\int_{y=x^2}^{4} F(x,y)\,dy\,dx = \int_{y=\dots}^{\dots}\int_{x=\dots}^{\dots} F(x,y)\,dx\,dy$$

2. Let R denote the region of points $(x,y) \in \mathbb{R}^2$ satisfying $x^2+y^2 \le 2$ and $x \ge 1$.

(a) Rewrite the following integral in polar coordinates:

$$I = \iint_{(x,y)\in R} F(x,y)\,dA = \int_{\theta=\dots}^{\dots}\int_{r=\dots}^{\dots} F(\dots,\dots)\dots$$

(b) Evaluate the integral from (a) for $F(x,y) = x$. *Hint:* $\frac{d}{dt}(\tan t) = 1/\cos^2 t$.

(c) Find the center of mass $(\bar{x},\bar{y})$ of the region R, assuming density 1.

3. Let D denote the tetrahedron with the vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$.

(a) Find $\iiint_{(x,y,z)\in D} x\,dV$

(b) Find the center of mass $(\bar{x},\bar{y},\bar{z})$ of the region D, assuming density 1.

4. Let D denote the region of points $(x,y,z) \in \mathbb{R}^3$ satisfying $x^2+y^2+z^2 \le 2$ and $z \ge \sqrt{x^2+y^2}$.

(a) Find the volume of D using cylindrical coordinates.

(b) Find the volume of D using spherical coordinates.

5. Let R denote the region of points $(x,y) \in \mathbb{R}^2$ satisfying $2x^{-1} \le y \le 3x^{-1}$ and $(1-x)^{-1} \le y \le 2(1-x)^{-1}$. Find the integral $\iint_{(x,y)\in R} y^2\,dA$. *Hint:* Use the variables $u = xy$ and $v = (1-x)y$.

6. Consider the cylindrical surface S consisting of points $(x,y,z) \in \mathbb{R}^3$ satisfying $x^2+z^2 = 4$.

(a) Let S_1 be the part of S with $-4 \le y \le 4$. Find a parametrization of S_1 and use it to compute the surface area.

(b) Let S_2 be the part of S with $0 \le z \le y$ and $y \le 4$. Find a parametrization of S_2 and use it to compute the surface area.

Solutions see next page

Solutions

1.

(a) $4-x^2 \geq 0 \Rightarrow -2 \leq x \leq 2$, so $R = \{(x,y) \mid -2 \leq x \leq 2, 0 \leq y \leq 4-x^2\}$
$I = \int_{x=-2}^{2} \int_{y=0}^{4-x^2} f(x,y)dy\,dx$
inner integral: $A(x) = \int_{y=0}^{4-x^2} (x^2+y)dy = \left[x^2 y + \frac{1}{2}y^2\right]_{y=0}^{4-x^2} = 8 - \frac{1}{2}x^4$
outer integral: $I = \int_{x=-2}^{2} (8-\frac{1}{2}x^4)dx = \left[8x - \frac{5}{2}x^5\right]_{x=-2}^{2} = \frac{128}{5}$

(b) $R = \{(x,y) \mid 0 \leq x \leq 2, x^2 \leq y \leq 4\}$. Smallest y-value: $y=0$ (for $x=0$), largest y-value: $y=4$, for given $y \in [0,4]$ find all y such that $(x,y) \in R$: must have $0 \leq x$ and $x^2 \leq y$, hence
$\int_{x=0}^{2} \int_{y=x^2}^{4} F(x,y)\,dy\,dx = \int_{y=0}^{4} \int_{x=0}^{\sqrt{y}} F(x,y)\,dx\,dy$

2.

(a) bounds for θ: $x = r\cos\theta \geq 1$, $r \leq \sqrt{2}$ imply $\cos\theta \geq 1/\sqrt{2} \iff \theta \in [-\pi/4, \pi/4]$
$x = r\cos\theta \geq 1 \iff r \geq 1/\cos\theta$, bounds for r: $1/\cos\theta \leq r \leq \sqrt{2}$

$$I = \int_{\theta=-\pi/4}^{\pi/4} \int_{r=1/\cos\theta}^{\sqrt{2}} F(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

(b) $M_1 = \int_{\theta=-\pi/4}^{\pi/4} \int_{r=1/\cos\theta}^{\sqrt{2}} r\cos\theta\, r\,dr\,d\theta$, $\int_{r=1/\cos\theta}^{\sqrt{2}} r^2\cos\theta\,dr = \cos\theta\left[\frac{1}{3}r^3\right]_{1/\cos\theta}^{\sqrt{2}} = \frac{2}{3}\sqrt{2}\cos\theta - \frac{1}{3}(\cos\theta)^{-2}$
$M_1 = \left[\frac{2}{3}\sqrt{2}\sin\theta - \frac{1}{3}\tan\theta\right]_{\theta=-\pi/4}^{\pi/4} = \frac{2}{3}$

(c) mass $M = \int_{\theta=-\pi/4}^{\pi/4} \int_{r=1/\cos\theta}^{\sqrt{2}} r\,dr\,d\theta = \frac{\pi}{2} - 1$, $M_2 = \int_{\theta=-\pi/4}^{\pi/4} \int_{r=1/\cos\theta}^{\sqrt{2}} r\sin\theta\, r\,dr\,d\theta = 0$ since integrand is anti-symmetric with respect to θ. Hence center of mass is $\bar{x} = M_1/M = \frac{2}{3}/\left(\frac{\pi}{2}-1\right)$, $\bar{y} = M_2/M = 0$.

3.

(a) $M_1 = \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{3-3x-(3/2)y} x\,dz\,dy\,dx = \frac{1}{4}$

(b) mass $M = \int_{x=0}^{1}\int_{y=0}^{2-2x}\int_{z=0}^{3-3x-(3/2)y} 1\,dz\,dy\,dx = 1$, $M_2 = \int_{x=0}^{1}\int_{y=0}^{2-2x}\int_{z=0}^{3-3x-(3/2)y} y\,dz\,dy\,dx = \frac{1}{2}$, $M_3 = \int_{x=0}^{1}\int_{y=0}^{2-2x}\int_{z=0}^{3-3x-(3/2)y} z$
$\frac{3}{4}$, center of mass: $\bar{x} = M_1/M = \frac{1}{4}$, $\bar{y} = M_2/M = \frac{1}{2}$, $\bar{z} = M_3/M = \frac{3}{4}$

4.

(a) $V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \frac{4}{3}\pi\left(\sqrt{2}-1\right)$

(b) $V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{\sqrt{2}} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{4}{3}\pi\left(\sqrt{2}-1\right)$

5. Solve $xy = u$ and $(1-x)y = v$ for x,y: $y = u+v$, $x = u/(u+v)$
Find Jacobian determinant:

$$\det\begin{bmatrix} \frac{\partial x}{\partial u}, & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v}, & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} \frac{v}{(u+v)^2}, & 1 \\ \frac{-u}{(u+v)^2}, & 1 \end{bmatrix} = \frac{1}{u+v}$$

$$\iint_{(x,y)\in R} y^2\,dA = \int_{u=2}^{3} \int_{v=1}^{2} (u+v)^2 \frac{1}{u+v}\,dv\,du = 4$$

6.

(a) parametrization: $x = 2\cos u$, $y = v$, $z = 2\sin u$, $0 \le u \le 2\pi$, $-4 \le v \le 4$
$\mathbf{r}_u \times \mathbf{r}_v = (-2\sin u, 0, 2\cos u) \times (0,1,0) = (-2\cos u, 0, -2\sin u)$, $\|\mathbf{r}_u \times \mathbf{r}_v\| = 2$

$$\int_{u=0}^{2\pi} \int_{v=-4}^{4} 2\, dv\, du = 32\pi$$

(b) parametrization: $x = 2\cos u$, $y = v$, $z = 2\sin u$, $0 \le u \le \pi$, $2\sin u \le v \le 4$

$$\int_{u=0}^{\pi} \int_{v=2\sin u}^{4} 2\, dv\, du = 8(\pi - 1)$$