Assignment #2 MATH246, Spring 2024
due 03/13 at 10pm

1. Do these problems with paper and pencil **without Matlab**.
   Find the **solution** of the initial value problem:

   (a) $y'' + y' - 2y = \sin t, \qquad y(0) = 1, \quad y'(0) = 2$
   Hint: for the homogeneous ODE try $y(t) = e^{rt}$, for the particular solution try $Y(t) = A\cos t + B\sin t$

   (b) $y'' + t^{-1}y' - t^{-2}y = t^2, \qquad y(1) = 1, \quad y'(1) = 1$
   Hint: for the homogeneous ODE try $y(t) = t^r$, for the particular solution try $Y(t) = At^s$

2. Consider the initial value problem $y'' + y' - 2y = t, \qquad y(0) = 1, \quad y'(0) = 1$

   (a) Find the solution with pencil and paper.

   (b) Perform one step of the Improved Euler method with $h = \frac{1}{2}$. What approximation do you get for $y(\frac{1}{2})$?

   (c) Write a Matlab function **`IEulerMethod`** for the Improved Euler method: modify the provided function `EulerMethod`. Use this to find the approximation Y for $y(2)$ using $n = 8, 16, 32, 64, 128$ steps. For each n print out n and the error $Y - y(2)$. Explain how you can see the convergence order in your results.

   (d) Use **`ode45`** to find an approximation Y for $y(8)$. Print the error $Y - y(8)$. Then use **`ode45`** with smaller tolerances **`RelTol`**, **`AbsTol`**. Try to obtain an error $|Y - y(8)| \le 10^{-10}$.

3. Consider the following ODEs:

$$\textbf{(i)}\ y'' + y' - 2y = 0, \qquad \textbf{(ii)}\ y'' + \tfrac{9}{4}y' + \tfrac{1}{2}y = 0$$

   (a) Rewrite the ODE as a 1st order system $\vec{y}\,' = \vec{f}(t, \vec{y})$. Use **`vectorfield`** to plot the vector field for $y_1, y_2 \in [-2, 2]$.
   For $a = -2, -1, 0, 1, 2$, for $b = -2, -1, 0, 1, 2$:
   Use **`ode45`** to solve the initial value problem with $y(0) = a,\ y'(0) = b$ first for $T = 8$ and then for $T = -8$.
   Plot all these trajectories in the phase plane as blue curves together with the vector field. At the end use **`axis([-2 2 -2 2])`**
   You should make one plot for (i) and one plot for (ii). Each plot should show the vector field together with 50 trajectories in the phase plane.

   (b) With paper and pencil: find the general solution of the ODE. Write down the general solution $\vec{y}(t) = c_1\vec{Y}^{(1)}(t) + c_2\vec{Y}^{(2)}(t)$ of the first order system. Consider the cases $c_1 = \pm 1$, $c_2 = 0$. Consider the cases $c_1 = 0$, $c_2 = \pm 1$. Explain what the corresponding trajectories in the phase plane look like, and how this corresponds to your "phase portrait" from (a).