Solutions of practice problems for Exam 2

1. Find the general solution of the ODE $y'' - 2y' + y = e^t$.
Solution: Find solutions of homogeneous problem: $p(r) = r^2 - 2r + 1 = 0$ gives $r_1 = r_2 = 1$, hence $y_1(t) = e^t$, $y_2(t) = te^t$.
For the particular solution use the method of undetermined coefficients: $Y(t) = At^2e^t$ since $r = 1$ is a double root of the characteristic polynomial $p(r)$. We get

$$Y'(t) = A2te^t + At^2e^t, \qquad Y''(t) = A2e^t + A4te^t + At^2e^t, \qquad Y'' - 2Y' + Y = 2Ae^t \stackrel{!}{=} e^t$$

Hence $A = \frac{1}{2}$. The general solution is $y(t) = Y(t) + c_1Y_1(t) + c_2Y_2(t) = \frac{1}{2}t^2e^t + c_1e^t + c_2te^t$.

2. Find the general solution of the ODE $y'' - 2y' + y = t^{1/2}e^t$.

Solution: Find solutions of homogeneous problem: $r^2 - 2r + 1 = 0$ gives $r_1 = r_2 = 1$, hence $y_1(t) = e^t$, $y_2(t) = te^t$. For the particular solution $Y(t)$ we cannot use the method of undetermined coefficients since $f(t)$ does not have the appropriate form. Therefore we must use variation of parameters:

$$\text{Wronskian: } W = y_1y_2' - y_1'y_2 = e^t(te^t + e^t) - e^tte^t = e^{2t}$$

$$u_1' = \frac{-y_2f}{W} = \frac{-te^tt^{1/2}e^t}{e^{2t}} = -t^{3/2}, \qquad u_1 = -\frac{2}{5}t^{5/2}$$

$$u_2' = \frac{y_1f}{W} = \frac{e^tt^{1/2}e^t}{e^{2t}} = t^{1/2}, \qquad u_2 = \frac{2}{3}t^{3/2}$$

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -\frac{2}{5}t^{5/2}e^t + \frac{2}{3}t^{3/2}te^t = \frac{4}{15}t^{5/2}e^t$$

Hence the general solution is $y(t) = \frac{4}{15}t^{5/2}e^t + c_1e^t + c_2te^t$.

3. We have a mass-spring-damper system with mass $m = 2$ and spring constant $k = 8$.

(a) Determine the damping constant c so that there is critical damping. Then determine the steady-state solution $Y(t)$ for the external force $F(t) = 2$.

Solution: The characteristic equation $2r^2 + cr + 8 = 0$ has solutions $r = \dfrac{-c \pm \sqrt{c^2 - 64}}{4}$. Hence we have critical damping for $c^2 = 64$ or $c = 8$. The steady state solution is the particular solution $Y(t)$: the method of undetermined coefficients gives $Y = A$ and the ODE then gives $8A = 2$, i.e., $Y(t) = A = 1/4$.

(b) Assume that there is no damping ($c = 0$) and that the external force is $F(t) = 2\sin(\omega t)$. Determine ω so that there is resonance.
Solution: The characteristic equation $2r^2 + 8 = 0$ has solutions $r = \pm 2i$, hence the solutions of the homogeneous ODE are $\cos(2t)$ and $\sin(2t)$. For the particular solution the method of undetermined coefficients gives $Y(t) = t^s[A\cos(\omega t)]$ with $s = 1$ for $\omega = 2$, $s = 0$ otherwise. Hence we have resonance for $\omega = 2$.

4.

(a) For the IVP

$$y'' + 3y' + 5y = f(t) = \begin{cases} t^2 & \text{for } t < 2 \\ 4 & \text{for } t \geq 2 \end{cases}, \qquad y(0) = 3, \quad y'(0) = -2$$

determine $Y(s)$ (the Laplace transform of the solution $y(t)$). DO **NOT** find $y(t)$. DO NOT simplify the result.
Solution: $f(t) = t^2 + u_2(t)\left(4 - t^2\right)$, $F(s) = \dfrac{2}{s^3} + e^{-2s}\mathcal{L}\{g(t+2)\}$ where $g(t) = 4 - t^2$. Hence $g(t+2) = 4 - (t+2)^2 = 4 - \left(t^2 + 4t + 4\right) = -t^2 - 4t$ and $F(s) = \dfrac{2}{s^3} + e^{-2s}\left(-\dfrac{2}{s^3} - \dfrac{4}{s^2}\right)$. Applying the Laplace transform to the ODE gives therefore

$$\left[s^2Y - s3 - (-2)\right] + 3\left[sY - 3\right] + 5Y = \frac{2}{s^3} + e^{-2s}\left(-\frac{2}{s^3} - \frac{4}{s^2}\right)$$

$$\left[s^2Y - s\cdot 3 - (-2)\right] + 3\left[sY - 3\right] + 5Y = \frac{2}{s^3} + e^{-2s}\left(-\frac{2}{s^3} - \frac{4}{s^2}\right)$$
$$Y(s) = \frac{1}{s^2+3s+5}\left[\frac{2}{s^3} + e^{-2s}\left(-\frac{2}{s^3} - \frac{4}{s^2}\right) + 3s + 2 + 9\right]$$

Note: Do not simplify this any further in the exam!

(b) Find the inverse Laplace transform $y(t)$ for

$$Y(s) = \frac{1+s}{s(s^2-2s+2)}$$

Solution: The zeros of the denominator are $s = 0, 1+i, 1-i$. Therefore the partial fraction decomposition is

$$\frac{1+s}{s\left[(s-1)^2+1^2\right]} = \frac{A}{s} + \frac{B + C(s-1)}{(s-1)^2+1^2}$$

Multiplying by the denominator gives

$$1 + s = A\left[(s-1)^2 + 1\right] + Bs + Cs(s-1)$$

Plugging in $s = 0$ and $s = 1$ gives

$$1 = A\cdot 2, \qquad A = \frac{1}{2}$$
$$2 = A + B, \qquad B = \frac{3}{2}$$

Now we can use an arbitrary s, e.g., $s = -1$ to get an equation for C: $0 = A\cdot 5 - B + C\cdot 2$, yielding $C = -\frac{1}{2}$ and

$$y(t) = \frac{1}{2} + \frac{3}{2}e^t\sin t + \frac{-1}{2}e^t\cos t$$

(c) Find the inverse Laplace transform $y(t)$ for

$$Y(s) = e^{-3s}\frac{4}{s^2(s^2-4)}$$

Solution: We first find $f(t) = \mathcal{L}^{-1}\{F(s)\}$ for $F(s) = \dfrac{4}{s^2(s^2-4)}$: The zeros of the denominator are $0, 0, 2, -2$, hence the partial fraction decomposition is

$$\frac{4}{s^2(s^2-4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{s+2}$$

Multiplying by the denominator gives

$$4 = As(s+2)(s-2) + B(s+2)(s-2) + Cs^2(s+2) + Ds^2(s-2)$$

Plugging in $s = 0, 2, -2$gives

$$4 = B\cdot(-4), \qquad B = -1$$
$$4 = C\cdot 16, \qquad C = \frac{1}{4}$$
$$4 = D\cdot(-16), \qquad D = -\frac{1}{4}$$

Now we can use an arbitrary s such as $s = 1$ to get an equation for A: $4 = A\cdot(-6) + B\cdot(-3) + C\cdot 3 + D\cdot(-1)$ which gives $A = 0$ and

$$f(t) = (-1)t + \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}.$$

Using $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$ we therefore get

$$y(t) = u_3(t)\left[-(t-3) + \frac{1}{4}e^{2(t-3)} - \frac{1}{4}e^{-2(t-3)}\right].$$