Linear ODE with constant coefficients: finding a fundamental set of solutions

ODE of order 2

We first consider the case of order 2. We have the homogeneous ODE

$$\underbrace{y'' + a_1 y' + a_0 y}_{Ly} = 0$$

with constants $a_0, a_1 \in \mathbb{R}$. We need to find solutions $Y_1(t), Y_2(t)$ which are **linearly independent** (i.e., we cannot write one function as a constant times the other function).

RECIPE: try to find solutions of the form $y(t) = e^{rt}$.

Then $y'(t) = re^{rt}$, $y''(t) = r^2 e^{rt}$, and plugging this into the ODE gives

$$\boxed{Le^{rt} = (r^2 + a_1 r + a_0)e^{rt}} \tag{1}$$

This is called "**KEY IDENTITY**". We want $Le^{rt} = 0$, so we must have

$$p(r) = r^2 + a_1 r + a_0 = 0$$

This is called the **characteristic equation**, and $p(r)$ is called the **characteristic polynomial**.

Here we get a quadratic equation. We know that $p(r) = (r - r_1)(r - r_2)$ with

$$r_1 = \tfrac{1}{2}\left(-a_1 + \sqrt{a_1^2 - 4a_0}\right)$$
$$r_2 = \tfrac{1}{2}\left(-a_1 - \sqrt{a_1^2 - 4a_0}\right)$$

For real coefficients $a_0, a_1 \in \mathbb{R}$ we have the following three cases:

Case 1: $\boxed{a_1^2 - 4a_0 > 0}$ We get two different real values $r_1 \neq r_2$, hence we get two solutions

$$\boxed{Y_1(t) = e^{r_1 t}, \qquad Y_2(t) = e^{r_2 t}}$$

Case 2: $\boxed{a_1^2 - 4a_0 = 0}$ We get two equal solutions $r_1 = r_2$, i.e., we have $p(r) = (r - r_1)^2$. Here $Y_1(t) = e^{r_1 t}$ is one solution. But we must find a second, linearly independent solution $Y_2(t)$. The key identity holds for all $r, t \in \mathbb{R}$

$$\boxed{(D^2 + a_1 D + a_0)e^{rt} = p(r)e^{rt}}$$

Here we consider e^{rt} as a function of r, t, and $D = \partial_t$. We now **take the partial derivative with respect to** r **on both sides**: On the left-hand side we get $\partial_r (D^2 + a_1 D + a_0)e^{rt} = (D^2 + a_1 D + a_0)\partial_r e^{rt}$ since $\partial_r \partial_t g(t,r) = \partial_t \partial_r g(t,r)$.

We have $\partial_r e^{rt} = te^{rt}$, hence we obtain the **DERIVATIVE OF THE KEY IDENTITY**

$$\boxed{L(te^{rt}) = p'(r)e^{rt} + p(r)te^{rt}}$$

In Case 2 we have $p(r) = (r - r_1)^2$, $p'(r) = 2(r - r_1)$ and

$$L(te^{rt}) = (r - r_1)^2 e^{rt} + 2(r - r_1)te^{rt}$$

With $r = r_1$ we obtain $L(te^{rt}) = 0$, so we found our second solution $Y_2(t) = te^{rt}$.

RECIPE: If we have two equal roots $r_1 = r_2$ we use $\boxed{Y_1(t) = e^{r_1 t}, \qquad Y_2(t) = te^{r_1 t}}$

Case 3: $\boxed{a_1^2 - 4a_0 < 0}$ In this case the quadratic formula gives us two complex roots

$$r_1 = \alpha + i\beta, \qquad r_2 = \alpha - i\beta \qquad \text{with } \alpha = -\tfrac{1}{2}a_1, \quad \beta = \tfrac{1}{2}\sqrt{4a_0 - a_1^2}$$

Note that

$$e^{it} = \cos t + i \sin t$$

which gives

$$\frac{e^{it}+e^{-it}}{2}=\cos t, \qquad \frac{e^{it}-e^{-it}}{2}=\sin t$$

The two solutions $\tilde{Y}_1(t)=e^{r_1 t}$, $\tilde{Y}_2(t)=e^{r_2 t}$ form a fundamental set of solutions, and the general solution is

$$y(t)=c_1\tilde{Y}_1(t)+c_2\tilde{Y}_2(t)$$

For initial conditions $y(t_0)=y_0$, $y'(t_0)=y_0'$ with $y_0, y_0' \in \mathbb{R}$ we will get $c_1, c_2 \in \mathbb{C}$, but the resulting function $y(t)=c_1\tilde{Y}_1(t)+c_2\tilde{Y}_2(t)$ will be real-valued.

It is easier to choose another fundamental set of solutions: we can use

$$Y_1(t)=\tfrac{1}{2}\left(\tilde{Y}_1(t)+\tilde{Y}_2(t)\right)=\tfrac{1}{2}\left(e^{(\alpha+i\beta)t}+e^{(\alpha-i\beta)t}\right)=e^{\alpha t}\tfrac{1}{2}\left(e^{i\beta t}+e^{-i\beta t}\right)=e^{\alpha t}\cos(\beta t)$$
$$Y_2(t)=\tfrac{1}{2i}\left(\tilde{Y}_1(t)-\tilde{Y}_2(t)\right)=\tfrac{1}{2i}\left(e^{(\alpha+i\beta)t}-e^{(\alpha-i\beta)t}\right)=e^{\alpha t}\tfrac{1}{2i}\left(e^{i\beta t}+e^{-i\beta t}\right)=e^{\alpha t}\sin(\beta t)$$

RECIPE: If we have two complex roots $r_1=\alpha+i\beta$ and $r_2=\alpha-i\beta$ we use $\boxed{Y_1(t)=e^{\alpha t}\cos(\beta t), \qquad Y_2(t)=e^{\alpha t}\sin(\beta t)}$

Example 1: $y''-5y'+6y=0$ gives the characteristic equation $r^2-5r+6=0$.
The roots are $r_1=2$, $r_2=3$, hence we get $Y_1(t)=e^{2t}$, $Y_2(t)=e^{3t}$.

Example 2: $y''-4y'+4y$ gives the characteristic equation $r^2-4r+4=0$.
The roots are $r_1=r_2=2$, hence we get $Y_1(t)=e^{2t}$, $Y_2(t)=te^{2t}$.

Example 3: $y''+y=0$ gives the characteristic equation $r^2+1=0$.
The roots are $r_1=i$, $r_2=-i$, hence we get $Y_1(t)=\cos t$, $Y_2(t)=\sin t$.

Example 4: $y''+4y+13y=0$ gives the characteristic equation $r^2+4r+13=0$.
The roots are $r_1=-2+3i$, $r_2=-2-3i$, hence we get $Y_1(t)=e^{-2t}\cos(3t)$, $Y_2(t)=e^{-2t}\sin(3t)$.

General case: ODE of order n

We have the homogeneous ODE

$$\underbrace{y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_0 y}_{Ly}=0$$

with constants $a_0,\ldots,a_{n-1}\in\mathbb{R}$. We need to find solutions $Y_1(t),\ldots,Y_n(t)$ which are **linearly independent** (i.e., we cannot write one function as a linear combination of the other functions).

RECIPE: try to find solutions of the form $y(t)=e^{rt}$.

Then $y'(t)=re^{rt}, \ldots, y^{(n)}(t)=r^n e^{rt}$, and plugging this into the ODE gives

$$Le^{rt}=(r^n+a_{n-1}r^{n-1}+a_0)e^{rt}$$

We call $p(t)=(r^n+a_{n-1}r^{n-1}+a_0)$ the **characteristic polynomial**. This gives the **KEY IDENTITY**

$$\boxed{Le^{rt}=p(r)e^{rt}} \tag{2}$$

Taking **partial derivatives of the key identity with respect to** r gives

$$\boxed{\begin{aligned} L\left(te^{rt}\right)&=p'(r)e^{rt}+p(r)te^{rt}\\ L(t^2e^{rt})&=p''(r)e^{rt}+2p'(rt)te^{rt}+p(r)t^2e^{rt}\end{aligned}} \tag{3}$$

STEP 1: Find the solutions $r_1,\ldots,r_n$ **of the equation** $p(r)=0$**.** According to the fundamental theorem of algebra there are roots $r_1,\ldots,r_n\in\mathbb{C}$ such that

$$p(r)=(r-r_1)\cdots(r-r_n)$$

We may not be able to find formulas for the roots (actually, for $n\geq 5$ one cannot in general express r_j in terms of $\sqrt{\cdot}$, $\sqrt[3]{\cdot}$ etc).

We have a polynomial with real coefficients $a_0,\ldots,a_{n-1}$. If we have $p(r)=0$ then we can take the complex conjugate and obtain $p(\overline{r})=0$ where $\overline{x+iy}=x-iy$ denotes the complex conjugate. Hence we have

- real roots
- pairs of complex conjugate roots $\alpha+i\beta$ and $\alpha-i\beta$

STEP 2: The fundamental set of solutions $Y_1(t), \ldots, Y_n(t)$ is given by the following functions:

- for a **simple real root** r use $\boxed{e^{rt}}$
- for a **real root** r **of multiplicity** m use $\boxed{e^{rt},\ te^{rt}, \ldots,\ t^{m-1}e^{rt}}$
- for **simple complex conjugate roots** $\alpha + i\beta$**,** $\alpha - i\beta$ use $\boxed{e^{\alpha t}\cos(\beta t),\ e^{\alpha t}\sin(\beta t)}$
- for **complex conjugate roots** $\alpha + i\beta$**,** $\alpha - i\beta$ **of multiplicity** m use
  $\boxed{e^{\alpha t}\cos(\beta t), \ldots, t^{m-1}e^{\alpha t}\cos(\beta t)}$ and $\boxed{e^{\alpha t}\sin(\beta t), \ldots, t^{m-1}e^{\alpha t}\sin(\beta t)}$

Justification for multiple roots: Assume e.g. that $p(r)$ has a triple root r_1, then we have

$$p(r) = (r - r_1)^3 q(r)$$

This implies $p(r_1) = 0$, $p'(r_1) = 0$, $p''(r_1) = 0$. Hence the key identities (2), (3) imply

$$Le^{r_1 t} = 0, \qquad L(te^{r_1 t}) = 0, \qquad L(t^2 e^{r_1 t}) = 0$$

Example: Consider the 7th order ODE $(D-1)^3(D^2+4D+13)^2 y = 0$.
The characteristic polynomial is $p(r) = (r-1)^3(r^2+4r+13)^2$.
We obtain a triple root 1 and double roots $-2+3i$, $-2-3i$.
Therefore we obtain the **fundamental set of the seven solutions**

$$\begin{gathered} e^t, \quad te^t, \quad t^2e^t, \\ e^{-2t}\cos(3t), \quad te^{-2t}\cos(3t), \\ e^{-2t}\sin(3t), \quad te^{-2t}\sin(3t) \end{gathered}$$