2nd order linear ODE with constant coefficients

The ODE

$$y'' + a_1 y'(t) + a_0 y(t) = g(t)$$

is called **homogeneous** if $g(t) = 0$, otherwise **inhomogeneous**. With $\boxed{D := \frac{d}{dt}}$ we can write this as

$$(D^2 + a_1 D + a_0)y = g$$

Note that the operator $L := (D^2 + a_1 D + a_0)$ is **linear**: we have for a functions u, v and $c \in \mathbb{R}$:

$$L(cu) = cLu, \qquad L(u+v) = Lu + Lv$$

Solving a homogeneous 2nd order linear ODE with constant coefficients

Example: Consider the initial value problem

$$\begin{aligned} y'' + 3y' + 2y &= 0 \\ y(0) = y_0, \quad y'(0) &= y'_0 \end{aligned} \tag{1}$$

with given values y_0, y'_0.

We try to find a solution of the form $\boxed{y(t) = e^{rt}}$: Plugging this in gives

$$(D^2 + 3D + 2)e^{rt} = (r^2 + 3r + 2)e^{rt} = 0$$

Since $e^{rt} \neq 0$ we must have

$$r^2 + 3r + 2 = 0$$

The quadratic formula gives two solutions:

$$r = \frac{-3 \pm \sqrt{9-8}}{2}, \qquad r_1 = -1, \quad r_2 = -2$$

Hence we get two solutions of the ODE (1)

$$\boxed{Y_1(t) = e^{-t}, \qquad Y_2(t) = e^{-2t}}$$

Note that also $c_1 e^{-t}$ and $c_2 e^{-2t}$ are solutions of $(D^2 + 3D + 2)y = 0$, and so is

$$\boxed{y(t) = c_1 Y_1(t) + c_2 Y_2(t)} \tag{2}$$

for any $c_1, c_2 \in \mathbb{R}$. Here

$$\begin{aligned} y(t) &= c_1 e^{-t} + c_2 e^{-2t} \\ y'(t) &= -c_1 e^{-t} - 2c_2 e^{-2t} \end{aligned}$$

For the initial value problem $y(0) = y_0,\ y'(0) = y'_0$ we plug in $t = 0$ and get

$$\begin{aligned} y(0) &= c_1 + c_2 = y_0 \\ y'(0) &= -c_1 - 2c_2 = y'_0 \end{aligned}$$

This is a linear system of two equations for two unknown values c_1, c_2. We can eliminate c_1 by adding the two equations:

$$-c_2 = y_0 + y'_0$$

which gives c_2. Then we can find c_1 from the first equation $c_1 = y_0 - c_2$.

Result: For any initial conditions $y(0) = y_0,\ y'(0) = y'_0$ we obtain a solution $y(t) = c_1 Y_1(t) + c_2 Y_2(t)$. We have uniqueness for the solution of the initial value problem. Hence **the general solution of the homogeneous ODE $Ly = 0$ is given by (2).**

Solving an inhomogeneous 2nd order linear ODE with constant coefficients

Example: Consider the initial value problem from the previous class

$$y'' + 3y' + 2y = t \tag{3}$$
$$y(0) = 2, \quad y'(0) = -1 \tag{4}$$

We first want to find some function $Y(t)$ satisfying $LY = g$ ("**particular solution**").
Here the right-hand side function $g(t)$ is a polynomial $g(t) = c_0 + c_1 t$.
If we apply the operator $L = D^2 + 3D + 2$ to a polynomial $Y(t) = C_0 + C_1 t$ we get again a polynomial $LY = c_0 + c_1 t$:

$$(D^2 + 3D + 2)Y = Y'' + 3Y' + 2Y = 0 + 3C_1 + 2(C_0 + C_1 t) = (2C_0 + 3C_1) + 2C_1 t$$

We want to have $LY = g = 0 + 1 \cdot t$, so we need to find C_0, C_1 satisfying

$$2C_0 + 3C_1 = 0$$
$$2C_1 = 1$$

The second equation gives $C_1 = \frac{1}{2}$, then the first equation gives $C_0 = -\frac{3}{2}C_1 = -\frac{3}{4}$. This gives the particular solution

$$Y(t) = -\tfrac{3}{4} + \tfrac{1}{2}t$$

satisfying $LY = g$.
Now assume that $y(t)$ is any function satisfying $Ly = g$. Taking the difference of the last two equations gives

$$L(y - Y) = 0,$$

i.e., the function $u(t) = y(t) - Y(t)$ satisfies the homogeneous ODE $Lu = 0$. But any solution of the homogeneous ODE must have the form

$$u(t) = c_1 Y_1(t) + c_2 Y_2(t)$$

yielding the **general solution of the inhomogeneous problem:**

$$\boxed{y(t) = Y(t) + c_1 Y_1(t) + c_2 Y_2(t)}$$

For the ODE (3) we get the general solution

$$y(t) = (-\tfrac{3}{4} + \tfrac{1}{2}t) + c_1 e^{-t} + c_2 e^{-2t}$$
$$y'(t) = \tfrac{1}{2} - c_1 e^{-t} - 2c_2 e^{-2t}$$

For the initial value problem $y(0) = 2,\ y'(0) = -1$ we plug in $t = 0$ and get

$$y(0) = -\tfrac{3}{4} + c_1 + c_2 = 2$$
$$y'(0) = \tfrac{1}{2} - c_1 - 2c_2 = -1$$

yielding the linear system

$$c_1 + c_2 = \tfrac{11}{4}$$
$$-c_1 - 2c_2 = -\tfrac{3}{2}$$

Adding the two equations gives $-c_2 = \frac{5}{4} \iff c_2 = -\frac{5}{4}$. Now the first equation gives $c_1 = \frac{11}{4} - c_2 = 4$. Hence the solution of the initial value problem (3), (4) is

$$y(t) = (-\tfrac{3}{4} + \tfrac{1}{2}t) + 4e^{-t} - \tfrac{5}{4}e^{-2t}$$